An Infinite Suite of Links–Gould Invariants

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February 1, 2008

Abstract

This paper describes a method to obtain state model parameters for an infinite series of Links–Gould link invariants $LG^{m,n}$, based on quantum R matrices associated with the $(\dot{0}_m \,|\, \dot{\alpha}_n)$ representations of the quantum superalgebras $U_q[gl(m|n)]$. Explicit details of the state models for the cases $n = 1$ and $m = 1, 2, 3, 4$ are supplied. Some gross properties of the link invariants are provided, as well as some explicit evaluations.

1 Overview

In 1992, Jon Links and Mark Gould [17] described a method for constructing link invariants from quantum superalgebras. That work stopped short of evaluations of the invariants due to want of an efficient computational method. In 1999, the author, in collaboration with Jon Links and Louis Kauffman [6], first evaluated a two-variable example of one these invariants, using a state model. We used the $(0, 0 \,|\, \alpha)$ representations of $U_q[gl(2|1)]$, and labeled our resulting $(1, 1)$-tangle invariant LG, '*the* Links–Gould invariant'. In that paper, and subsequently in [4], we showed that whilst LG would detect neither inversion nor mutation, it was still able to distinguish all prime knots of up to 10 crossings, making it more powerful than the HOMFLY and Kauffman invariants.

Here, we generalise the notation, denoting $LG^{m,n}$ as "the Links–Gould invariant associated with the $(\dot{0}_m \,|\, \dot{\alpha}_n)$ representation of $U_q[gl(m|n)]$". For the case $n = 1$, we will write $LG^m \equiv LG^{m,1}$, so our previous invariant LG was in fact LG^2. This generalisation is motivated by the automation of a procedure to construct the appropriate R matrices [3, 5]; previously, we were limited to the $m = 2$ case, for which the R matrix had been calculated by hand.

We explicitly demonstrate the construction of state model parameters for $LG^{m,n}$, illustrating our results for LG^m, for the cases $m = 1, 2, 3, 4$. Further, we describe some of the gross properties of these invariants, and provide a limited set of evaluations of them.

Although these invariants $LG^{m,n}$ are not more powerful in their gross properties than LG^2 (they can detect neither inversion nor mutation), each one is expected to distinguish many more knots K as the degree of the polynomials $LG^{m,n}_K$ increases rapidly with m and n. Perhaps more significantly, the development of the current formalism points the way towards automation of the evaluation of more general classes of quantum link invariants; a discussion of this is provided.

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2 Quantum superalgebra state models

Corresponding to each finite dimensional highest weight representation of each quantum superalgebra, there exists a quantum link invariant (LG), originally described in [17]. These invariants are similar to those associated with the usual (i.e. ungraded) quantum algebras (e.g. [10, 11, 23]), although there are some technical differences.

Here, we describe the construction of parameters for state models for evaluating a class of these invariants. Specifically, we will define $LG^{m,n}$ to be the quantum link invariant associated with the representation $\pi \equiv \pi_\Lambda$ of highest weight $\Lambda = (\dot{0}_m \mid \dot{\alpha}_n)$, of the quantum superalgebra $U_q[gl(m|n)]$. To do so, we first broadly introduce the algebraic structures, then we briefly review the terminology used to describe state model parameters, and finally, we look at the construction of specific state model parameters for our particular class of representations.

2.1 The quantum superalgebra $U_q[gl(m|n)]$

$U_q[gl(m|n)]$ is a unital super (i.e. $\mathbb{Z}_2$-graded) algebra with free parameter q. In the limit $q \to 1$, it degenerates to the ordinary Lie superalgebra $gl(m|n)$. Here, we provide a broad outline of $U_q[gl(m|n)]$ in terms of generators and relations, for readers not familiar with it. This material is largely abstracted from the fuller description contained in [25] (see also [3]).

2.1.1 $U_q[gl(m|n)]$ generators

A set of generators for $U_q[gl(m|n)]$ is:

$$\left\{ \begin{array}{lll} K_a, & 1 \leqslant a \leqslant m+n & \text{Cartan} \\ E^a{}_b, & 1 \leqslant a < b \leqslant m+n & \text{raising} \\ E^b{}_a, & 1 \leqslant a < b \leqslant m+n & \text{lowering} \end{array} \right\}.$$

An equivalent notation for K_a is $q_a^{E^a{}_a}$, where we have introduced the notation $q_a \triangleq q^{(-)^{[a]}}$. For any power N, we may write q_a^N, and hence K_a^N, thus for $M, N \in \mathbb{C}$:

$$K_a^M K_a^N = K_a^{M+N} \quad \text{where} \quad K_a^0 \equiv \text{Id},$$

where Id is the $U_q[gl(m|n)]$ identity element.

Using the following $\mathbb{Z}_2$ grading on the $gl(m|n)$ *indices*:

$$[a] \triangleq \left\{ \begin{array}{llll} 0 & \text{if} & 1 \leqslant a \leqslant m & \text{even} \\ 1 & \text{if} & m+1 \leqslant a \leqslant m+n & \text{odd,} \end{array} \right.$$

we may define a natural $\mathbb{Z}_2$ grading on the generators:

$$[K_a^N] \triangleq 0, \qquad [E^a{}_b] \triangleq [a] + [b] \pmod 2,$$

and we use the terms "even" and "odd" for generators in the same manner as we do for indices. Elements of $U_q[gl(m|n)]$ are said to be *homogeneous* if they are linear combinations of generators of the same grading. The product XY of homogeneous $X, Y \in U_q[gl(m|n)]$ has grading:

$$[XY] \triangleq [X] + [Y] \pmod 2.$$

Within the full set of generators, we have the $U_q[gl(m|n)]$ *simple* generators:

$$\left\{\begin{array}{rll} K_a, & 1 \leqslant a \leqslant m+n & \text{Cartan} \\ {E^a}_{a+1}, & 1 \leqslant a < m+n & \text{simple raising} \\ {E^{a+1}}_a, & 1 \leqslant a < m+n & \text{simple lowering} \end{array}\right\},$$

such that the remaining nonsimple generators may be expressed in terms of these [25, p1238, (2)]. The fact that there are $m+n-1$ simple raising generators indicates that $U_q[gl(m|n)]$ has rank $m+n-1$.

2.1.2 $U_q[gl(m|n)]$ relations

The *graded commutator* $[\cdot,\cdot] : U_q[gl(m|n)] \times U_q[gl(m|n)] \to U_q[gl(m|n)]$, is defined for homogeneous $X, Y \in U_q[gl(m|n)]$ by:

$$[X, Y] \triangleq XY - (-)^{[X][Y]} YX,$$

and extended by linearity. With this, we have the following $U_q[gl(m|n)]$ relations:

1. The Cartan generators all commute:

$$K_a^M K_b^N = K_b^N K_a^M, \qquad M, N \in \mathbb{C}.$$

2. The Cartan generators commute with the simple raising and lowering generators in the following manner:

$$K_a {E^b}_{b\pm1} = q_a^{(\delta_b^a - \delta_{b\pm1}^a)} {E^b}_{b\pm1} K_a.$$

3. The squares of the odd simple generators are zero:

$$\left({E^m}_{m+1}\right)^2 = \left({E^{m+1}}_m\right)^2 = 0.$$

   (This implies that the squares of nonsimple odd generators are also zero.)

4. The non-Cartan generators satisfy the following commutation relations:

$$[{E^a}_{a+1}, {E^{b+1}}_b] = \delta_b^a \frac{K_a K_{a+1}^{-1} - K_a^{-1} K_{a+1}}{q_a - \overline{q}_a},$$

   where we have written $\overline{q} \equiv q^{-1}$ for brevity. We also have, for $|a-b| > 1$, the commutations:

$${E^a}_{a+1} {E^b}_{b+1} = {E^b}_{b+1} {E^a}_{a+1} \qquad \text{and} \qquad {E^{a+1}}_a {E^{b+1}}_b = {E^{b+1}}_b {E^{a+1}}_a.$$

5. Lastly, we have the $U_q[gl(m|n)]$ *Serre relations*; their inclusion ensures that the algebra is reduced enough to be *simple*. We omit these for brevity; they are not required below.

2.1.3 $U_q[gl(m|n)]$ as a Hopf superalgebra

When equipped with an appropriate[1] coproduct Δ, counit ε and antipode S, we may regard $U_q[gl(m|n)]$ as a quasitriangular Hopf superalgebra. This means that it possesses an R matrix $\check{R}$, an operator on the tensor product $U_q[gl(m|n)] \otimes U_q[gl(m|n)]$, satisfying the quantum Yang–Baxter equation (QYBE) in the form:

$$(\check{R} \otimes I)(I \otimes \check{R})(\check{R} \otimes I) = (I \otimes \check{R})(\check{R} \otimes I)(I \otimes \check{R}), \tag{1}$$

immediately recognisable as the braid relation:

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2. \tag{2}$$

2.2 State model parameters

The following comments briefly describe in what is well-documented in the literature (see, e.g. [8, 12]). They are included so as to introduce our particular notation.

A *state model* (σ, C) for a link invariant consists of two parameters: σ, an invertible rank 4 tensor representing the braid generator (i.e. a positive crossing), and C, an invertible rank 2 tensor representing a *positive handle*, that is an anticlockwise-oriented, vertical open arc, used to close a one string of a braid. From these, we may immediately define the representations corresponding to negative crossings (viz $\overline{\sigma} \equiv \sigma^{-1}$) and negative handles (viz $\overline{C} \equiv C^{-1}$).[2] Our current collection of arcs is shown in Figure 1.

Figure 1: Diagram components corresponding to σ and C.

Let β be a braid corresponding to a link $L \equiv \hat{\beta}$, formed from the vertical closure of β. The diagram components corresponding to σ (and $\overline{\sigma}$) are sufficient to construct β, and those corresponding to C (we don't need $\overline{C}$) are then sufficient to construct $\hat{\beta}$ from β.

When the pair (σ, C) is chosen to satisfy the Reidemeister moves (below, we write R1, R2 and R3), we may form a link invariant from the contraction over the free indices of the tensors corresponding to the diagram components. It may happen that the algebraic structures underlying the model mean that this invariant will be zero on closed links (i.e. $(0,0)$-tangles), however, we may still form an invariant of $(1,1)$-tangles [1, 6], by contracting over all but one free index, and obtain an invariant which is not necessarily trivial. Our invariants LG are based on typical $U_q[gl(m|n)]$ representations, for which the appropriate supertrace is zero, hence we define our invariants to be $(1,1)$-tangle invariants.

[1]The details of these structures are not required here; the reader can find them in [2, 3].

[2]We frequently use the notation $\overline{X}$ to mean X^{-1}, in particular, writing $\overline{\sigma} \equiv \sigma^{-1}$ and $\overline{C} \equiv C^{-}$ allows us to omit superfluous "+" signs, viz we write $C \equiv C^{+}$ for the positive handle.

2.3 State model parameters for $U_q[gl(m|n)]$ representations Λ

Here, we integrate the materials of §2.1 and §2.2, allowing us to describe the construction of state model parameters corresponding to arbitrary $U_q[gl(m|n)]$ representations Λ. Below, in §2.4, we perform some extra necessary calculations. After that, in §3, we specialise this material to the case $\Lambda = (\dot{0}_m \mid \dot{\alpha}_n)$.

So, how do we construct state model parameters (σ, C) corresponding to an invariant associated with an arbitrary $U_q[gl(m|n)]$ representation π?

Firstly, the tensor product representation $\check{R} \equiv (\pi \otimes \pi)\check{R}$ necessarily satisfies the QYBE in the form (1), and hence the braid relation (2). This means that abstract tensors built from $\check{R}$ are invariant under R2 and R3, hence we may construct representations of arbitrary braids from $\check{R}$. Thus $\sigma \triangleq \kappa_\sigma \check{R}$ (for any scalar constant κ_σ) realises a representation of the braid generator.

A technical point distinguishes the quantum superalgebra situation from that of the quantum algebra. Quantum superalgebra R matrices are in fact *graded*, and actually satisfy a *graded* QYBE. It is however, a simple matter to strip out this grading (i.e. apply an automorphism [5]),[3] yielding $\check{R}$ satisfying the usual, ungraded QYBE.

Secondly, to ensure that our invariant is an invariant of ambient isotopy, we must select C to ensure that abstract tensors built from σ and C are also invariant under R1. To this end, we apply (a grading-stripped version of) the following result [19, Lemma 2] (see also [17]):

$$(I \otimes \mathrm{str})[(I \otimes q^{2h_\rho})\sigma] = KI,$$

where the Cartan element q^{h_ρ} is defined in §2.4, str is the supertrace, and K is some constant depending on the normalisations of σ and q^{h_ρ}. Writing $S \equiv \pi(q^{2h_\rho})$ for convenience; for any scalar constant κ_C, setting $C \triangleq \kappa_C S$ allows us to represent positive handles. It remains to choose (κ_S, κ_C) to satisfy R1.

Thus, we demonstrate how to select κ_σ and κ_C such that the abstract tensor associated with removal of an isolated loop is invariant under R1. Figure 2 shows that for σ and C to satisfy R1, they must satisfy (Einstein summation convention):

$$C^d_c \cdot \sigma^{ca}_{db} = \delta^a_b = C^d_c \cdot \overline{\sigma}^{ca}_{db}, \tag{3}$$

where the definitions of κ_σ and κ_C yield: $\overline{\sigma} = \kappa_\sigma^{-1}\check{R}^{-1}$, and $\overline{C} = \kappa_C^{-1}S^{-1}$.

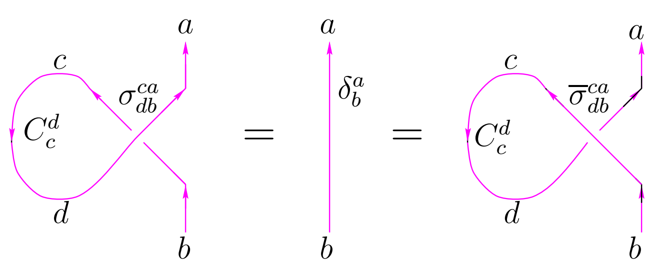


Figure 2: The first Reidemeister move.

[3]The $\check{R}$ supplied in [5] are normalised such that $\lim_{q\to 1} \check{R}$ is a (graded) permutation matrix. Scaling by κ_σ does not change that.

So, if we have established S and $\check{R}$, we may determine κ_σ and κ_C by solving the following equations:

$$\kappa_C \kappa_\sigma S^d_c \cdot (\check{R})^{ca}_{db} = \kappa_C \kappa_\sigma^{-1} S^d_c \cdot (\check{R}^{-1})^{ca}_{db} = \delta^a_b.$$

Setting $a = b = 1$ and using the fact that S is diagonal, we thus have:

$$\kappa_\sigma = X_1^{-\frac{1}{2}} X_2^{+\frac{1}{2}}, \qquad \kappa_C = X_1^{-\frac{1}{2}} X_2^{-\frac{1}{2}},$$

where $X_1 \triangleq S^c_c \cdot (\check{R})^{c1}_{c1}$ and $X_2 \triangleq S^c_c \cdot (\check{R}^{-1})^{c1}_{c1}$.

Note that reflecting the diagrams of Figure 2 about a vertical axis yields exactly the same constraints on κ_σ and κ_C. To see this, the constraints obtained by reflecting the diagrams in a vertical axis are:

$$\overline{\sigma}^{ac}_{bd} \cdot \overline{C}^d_c = \delta^a_b = \sigma^{ac}_{bd} \cdot \overline{C}^d_c, \tag{4}$$

however, we have: $\overline{\sigma}^{ac}_{bd} = (\sigma^{ca}_{db})|_{q \mapsto \overline{q}}$ and $\overline{C}^d_c = (C^d_c)|_{q \mapsto \overline{q}}$. Replacing $q \mapsto \overline{q}$ in (4) and applying these equivalences recovers (3). Similarly, reversing the orientations of the strings in Figure 2 yields no new constraints.

What is significant in the above is that we have explicit formulae for automatically scaling from $(\check{R}, S)$ to (σ, C), something apparently absent in the literature. Variations on these formulae should hold for a much wider class of representations and algebraic structures. We write them up as a little lemma:

Lemma 1 *Let π be a finite-dimensional highest weight $U_q[gl(m|n)]$ representation, for which we have computed $\check{R} \equiv (\pi \otimes \pi)\check{R}$ and $S \equiv \pi(q^{2h_\rho})$. Then the state model parameters (σ, C) for the corresponding link invariant of ambient isotopy may be obtained from $(\check{R}, S)$ by the scalings $\sigma = (X_1^{-1} X_2)^{\frac{1}{2}} \check{R}$ and $C = (X_1^{-1} X_2^{-1})^{\frac{1}{2}} S$, where $X_1 \triangleq S^c_c \cdot (\check{R})^{c1}_{c1}$ and $X_2 \triangleq S^c_c \cdot (\check{R}^{-1})^{c1}_{c1}$.*

2.3.1 Negative Handles, Caps and Cups

Demanding that our model parameters satisfy R0 (ambient isotopy in the plane) allows us to determine appropriate values for negative handles, caps and cups (see [4]). Although we *can* evaluate our invariants without these, we describe them here for completeness and backwards compatibility.

Firstly, the negative handle $\overline{C}$ is simply $C|_{q \mapsto \overline{q}}$. Secondly, although there is some flexibility in the choice of suitable caps $\Omega^\pm$ and cups $\mho^\pm$, in fact it is natural to choose them to be the square roots of the handles $C^\pm$:

$$\Omega^\pm = \mho^\pm = (C^\pm)^{\frac{1}{2}}, \tag{5}$$

taking the positive square root by convention. Note that these choices further improve those of our previous work [4, 6] by increasing the symmetries between the diagram components.

Satisfaction of R0 is described in Figure 3, that is, we demand:

$$\overline{\Omega}_{bc} \cdot \mho^{ca} = \delta^a_b = \overline{\mho}^{ac} \cdot \Omega_{cb}. \tag{6}$$

The definition (5) ensures that (6) is satisfied. In fact, the LHS and RHS of (6) are actually equivalent, hence one is redundant. Again, reversing the orientations of the strings in Figure 3 yields no new constraints.

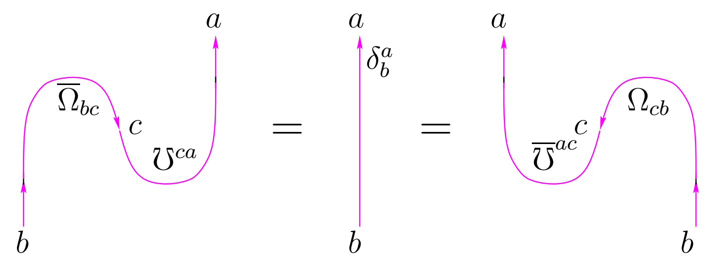


Figure 3: The zeroth Reidemeister move.

2.4 q^{h_ρ} for $U_q[gl(m|n)]$

Here, we determine the form that q^{h_ρ} takes in $U_q[gl(m|n)]$, in terms of Cartan generators. Recall that for any particular representation π, our state model requires (a grading-stripped version of) $S = \pi(q^{2h_\rho})$, and this may be obtained by substitution of the appropriate matrix elements into the expression for q^{2h_ρ}.

Initially, we shall work with $gl(m|n)$. To this end, let H be the Cartan subalgebra of $gl(m|n)$, with dual the root space H^*. A basis for H^* is given by the *fundamental weights* $\{\varepsilon_i\}_{i=1}^{m+n}$, which are elementary unit vectors of $m+n$ components, with 1 in position i and 0 elsewhere. On H^*, we have the following invariant bilinear form $(\cdot,\cdot) : H^* \times H^* \to \mathbb{C}$:

$$(\varepsilon_i, \varepsilon_j) \triangleq (-)^{[i]}\delta_{ij}, \tag{7}$$

and as H and H^* are dual, we of course have the form:

$$E^j{}_j(\varepsilon_i) \triangleq \delta_{ij}, \tag{8}$$

for $gl(m|n)$ Cartan generators $E^j{}_j$, $j = 1, \ldots, m+n$.

To the $gl(m|n)$ root $\varepsilon_i - \varepsilon_j$, there corresponds a $gl(m|n)$ Chevalley generator $E^i{}_j$, and we assign a grading and a sign to the roots in accordance with those of these generators.

In terms of these, $gl(m|n)$ has the following simple, positive roots:

$$\alpha_i \triangleq \varepsilon_i - \varepsilon_{i+1}, \qquad i = 1, \ldots, m+n-1, \tag{9}$$

in the sense that these form a basis for H^*. Apart from the single odd root α_m, the simple positive roots are all even. (Of various choices for superalgebra root systems, this *distinguished root system* is unique in containing only one odd root.)

Where Δ^+ is the set of *all* positive roots, and γ denotes the grading of the root γ, we define ρ as the graded half sum of all positive roots: $\rho \triangleq \frac{1}{2}\sum_{\gamma\in\Delta^+}(-)^{[\gamma]}\gamma$. Explicitly, for $gl(m|n)$, we have [7, p6207]:

$$\rho = \tfrac{1}{2}\sum_{i=1}^{m}(m-n-2i+1)\varepsilon_i + \tfrac{1}{2}\sum_{i=m+1}^{m+n}(3m+n-2i+1)\varepsilon_i,$$

although we will not actually require this form.

We are actually interested in $h_\rho \in gl(m|n)$, defined to satisfy:

$$h_\rho(\alpha_i) \triangleq (\rho, \alpha_i), \qquad \forall \alpha_i, \tag{10}$$

where we intend (8) on the LHS and (7) on the RHS. From the definition of ρ:

$$(\rho, \alpha_i) = \tfrac{1}{2}(\alpha_i, \alpha_i) \stackrel{(9)}{=} \tfrac{1}{2}(\varepsilon_i - \varepsilon_{i+1}, \varepsilon_i - \varepsilon_{i+1}) \stackrel{(7)}{=} \tfrac{1}{2}[(-)^{[i]} + (-)^{[i+1]}]. \tag{11}$$

As h_ρ is a Cartan element of $gl(m|n)$, we may express it as a linear combination of Cartan generators ${E^i}_i$; viz for some undetermined scalar coefficients β_i, we may set: $h_\rho = \sum_{i=1}^{m+n} \beta_i {E^i}_i$. Substituting this into (10) yields:

$$h_\rho(\alpha_i) = \sum_{j=1}^{m+n} \beta_j {E^j}_j(\varepsilon_i - \varepsilon_{i+1}) \stackrel{(9,8)}{=} \beta_i - \beta_{i+1}. \tag{12}$$

Substituting (11) and (12) into (10), we have:

$$\beta_i - \beta_{i+1} = \begin{cases} +1 & i = 1, \ldots, m-1 \\ 0 & i = m \\ -1 & i = m+1, \ldots, m+n-1. \end{cases} \tag{13}$$

For symmetry, selecting $\beta_m = \theta$ and substituting backwards and forwards yields:

$$\beta_i = \theta + \begin{cases} m - i & i = 1, \ldots, m \\ i - (m+1) & i = m+1, \ldots, m+n, \end{cases}$$

therefore:

$$\begin{aligned} h_\rho &= \sum_{i=1}^{m+n} \beta_i {E^i}_i = \sum_{i=1}^{m} (\theta + m - i){E^i}_i + \sum_{i=m+1}^{m+n} (\theta + i - (m+1)){E^i}_i \\ &= \theta C_1 + \sum_{i=1}^{m} (m-i){E^i}_i + \sum_{i=m+1}^{m+n} (i - (m+1)){E^i}_i, \end{aligned}$$

where $C_1 \triangleq \sum_{i=1}^{m+n} {E^i}_i$ is the first-order Casimir element of $gl(m|n)$. This shows us that h_ρ is only determined up to an additive constant.[4]

In passing from $gl(m|n)$ to $U_q[gl(m|n)]$, we pass from h_ρ to q^{h_ρ}, hence we have:

$$\begin{aligned} q^{h_\rho} &= q^{\theta C_1} \cdot q^{\sum_{i=1}^{m}(m-i){E^i}_i} \cdot q^{\sum_{i=m+1}^{m+n}(i-(m+1)){E^i}_i}, \\ &= \left(q^{C_1}\right)^\theta \cdot \prod_{i=1}^{m} \left(q^{{E^i}_i}\right)^{m-i} \cdot \prod_{i=m+1}^{m+n} \left(q^{{E^i}_i}\right)^{i-(m+1)} \\ &= \left(q^{C_1}\right)^\theta \cdot \prod_{i=1}^{m} K_i^{m-i} \cdot \prod_{i=m+1}^{m+n} K_i^{(m+1)-i}, \end{aligned}$$

where we have reminded ourselves of the definition $K_i \triangleq q^{(-)^{[i]}{E^i}_i}$. Thus, q^{h_ρ} is only determined up to an arbitrary multiplicative constant. Selecting $\theta = 0$, we declare the resulting product to be the standard q^{h_ρ}. For arbitrary m, n, we have:

$$q^{h_\rho} = K_1^{m-1} K_2^{m-2} \cdots K_{m-1}^{1} K_m^{0} \cdot K_{m+1}^{0} K_{m+2}^{-1} \cdots K_{m+n}^{-(n-1)}, \tag{14}$$

where of course K_i^0 is the $U_q[gl(m|n)]$ identity element.

[4]For $sl(m|n)$ and $sl(n)$, h_ρ is actually unique. C_1 also satisfies $C_1(\alpha_i) = 0, \forall \alpha_i$.

For our state models we require $S = \pi(q^{2h_\rho})$. To construct S, it suffices to compute matrix elements for the $U_q[gl(m|n)]$ Cartan generators K_i, and insert (appropriate powers of) these into (14), finally stripping the grading from S. In [5], we described the automation of the construction of $\check{R}$ corresponding to the $U_q[gl(m|1)]$ representations $(\dot{0}_m \,|\, \alpha)$, for arbitrary m, and obtained explicit $\check{R}$ for $m = 1, 2, 3, 4$. Explicit matrix elements for the K_i are obtained as a byproduct of that construction, facilitating the evaluation of S.

What is particularly interesting about this work is that the entire process, from the construction of the underlying representations [3, 5], to the scaling of the state model parameters, to the final evaluations of the polynomials, has been automated. This represents a step forward in computational power in knot theory.

3 The quantum link invariants $LG^{m,n}$

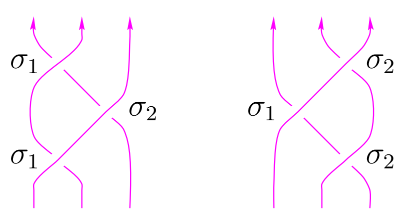
Having described the construction of state models for arbitrary finite dimensional highest weight $U_q[gl(m|n)]$ representations Λ, we now restrict our attention to the case:

$$\Lambda = (\dot{0}_m \mid \dot{\alpha}_n) \equiv (0, \ldots, 0 \mid \alpha, \ldots, \alpha),$$

and the resulting invariants $LG^{m,n}$. Evaluation of $LG^{m,n}$ for any particular link follows from that for LG^2, described in our previous work [4, 6]. Below, we make a few comments on the properties of $LG^{m,n}$, before describing in §4 some computational issues and evaluations for LG^3 and LG^4.

3.1 Checking the QYBE and applying the Matveev Δ–∇ test

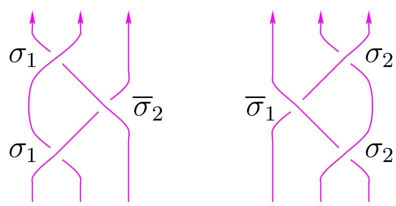
To be certain that we have made no errors in our computations, we check that our braid generator σ satisfies the (quantum) Yang–Baxter equation. The code used to construct the tensors Z_K is immediately adaptable to such a test. If Z is the same for the braids $\sigma_1\sigma_2\sigma_1$ and $\sigma_2\sigma_1\sigma_2$, then our braid generator satisfies the QYBE. This is depicted in Figure 4.



$$Z_{\sigma_1\sigma_2\sigma_1} \stackrel{?}{=} Z_{\sigma_2\sigma_1\sigma_2}$$

Figure 4: Checking that σ satisfies the QYBE.

The same framework allows us to carry out a simple sufficiency check to determine if a link invariant associated with some R matrix solution of the QYBE will be trivial.[5] Matveev [20] (see also [22]) introduced a 'delta unknotting operation' (which we call the Matveev Δ–∇ test), and proved that any knot can be transformed to the unknot by using only this operation. In our tensor language, if Z fails to distinguish $\sigma_1\overline{\sigma}_2\sigma_1$ and $\sigma_2\overline{\sigma}_1\sigma_2$, then the associated invariant will be trivial, as a series of exchanges of crossings of this form is always sufficient to convert any links to the unknot. Matveev's test is depicted in Figure 5.



$$Z_{\sigma_1\overline{\sigma}_2\sigma_1} \stackrel{?}{=} Z_{\sigma_2\overline{\sigma}_1\sigma_2}$$

Figure 5: The Matveev Δ–∇ test.

Both these tests have been satisfactorily carried out for our various braid generators σ, viz each σ satisfies the QYBE and the invariant built from it is not necessarily trivial.

3.2 Behaviour of $LG^{m,n}$ under inversion of q

Let K^* denote the reflection of a link K. In [6], we showed that $LG^2_{K^*} = LG^2_K|_{q\to\overline{q}}$. This result immediately carries over to $LG^{m,n}$, and means that if $LG^{m,n}_K$ is palindromic in q (i.e. invariant under the inversion $q \to \overline{q}$), then $LG^{m,n}$ cannot distinguish the chirality of K. Examples illustrating that LG^2 can distinguish the chirality of all prime knots of up to 10 crossings [4] demonstrate that $LG^{m,n}$ can indeed sometimes distinguish chirality, although counterexamples are expected to exist.

3.3 $LG^{m,n}$ doesn't detect mutation

Theorem 5 of [21] shows that quantum invariants based on R matrices where the orthogonal decomposition of $V \otimes V$ contains no multiplicities will not distinguish mutants. The extension of this result to quantum superalgebras is straightforward, and as our invariants $LG^{m,n}$ are indeed based on representations of this type [7], they will not distinguish mutants.

[5] This test is known to be a sufficient (but perhaps not a necessary) test of triviality – it doesn't even guarantee the *existence* of an invariant.

3.4 Behaviour of $LG^{m,n}$ under representation duality

In [6, Proposition 3.2], we showed that link invariants derived from irreducible representations of quantum (super)algebras are unable to detect knot inversion, as such invariants are necessarily equivalent to invariants constructed from their dual representations. Let us determine what this means for $LG^{m,n}$.

Let $V \equiv V_\Lambda$ be the module associated with $\pi \equiv \pi_\Lambda$, viz V has a highest weight vector v_+, of weight Λ. The corresponding lowest weight vector of V is obtained by the combined action of all the odd lowering operators: $\prod E^{m+j}{}_i$ on v_+, where the product is over all $i = 1, \ldots, m$ and $j = 1, \ldots, n$. The action of $E^{m+j}{}_i$ on a weight vector lowers its weight by $\varepsilon_i - \varepsilon_{m+j}$, viz:

$$(0, \ldots, 0, 1, 0, \ldots, 0 \mid 0, \ldots, 0, -1, 0, \ldots, 0).$$

The resulting lowest weight vector of V thus has weight $\overline{\Lambda}$:

$$\begin{aligned} \overline{\Lambda} &= \Lambda - \textstyle\sum_{ij}(\varepsilon_i - \varepsilon_{m+j}) = \Lambda - n\sum_{i=1}^{m}\varepsilon_i - m\sum_{j=1}^{n}\varepsilon_{m+j} \\ &= (-n, \ldots, -n \mid \alpha + m, \ldots, \alpha + m). \end{aligned}$$

The dual of V is labeled V^*, and naturally has highest weight $-\overline{\Lambda}$:

$$-\overline{\Lambda} = (n, \ldots, n \mid -\alpha - m, \ldots, -\alpha - m),$$

but V^* is equivalent to the module of highest weight Λ^*:

$$\Lambda^* = (0, \ldots, 0 \mid -\alpha - (m-n), \ldots, -\alpha - (m-n)),$$

hence we may regard the representations Λ and Λ^* as duals. Thus, at least up to a scalar multiple, we expect $LG^{m,n}$ to be invariant under the transformation $\alpha \mapsto -\alpha - (m-n)$, equivalently in the more symmetric form: $\alpha + \frac{m-n}{2} \mapsto -\alpha - \frac{m-n}{2}$, viz:

$$LG^{m,n}(q, q^{\alpha + \frac{m-n}{2}}) = LG^{m,n}(q, q^{-\alpha - \frac{m-n}{2}}).$$

Thus, if we define:

$$p \triangleq q^{\alpha + \frac{m-n}{2}}, \tag{15}$$

we have, again, up to a scalar multiple, the symmetry:

$$LG^{m,n}(q, p) = LG^{m,n}(q, \overline{p}). \tag{16}$$

Experiments show that the scalar multiple is always ± 1, and, for knots, always 1. Where $\overline{K}$ is the inverse of a knot K, inspection of diagram components shows that $LG^{m,n}_{\overline{K}}(q, p) = LG^{m,n}_K(q, \overline{p})$, hence (16) shows that $LG^{m,n}$ is unable to detect the inversion of knots.

Experimentally (setting $n = 1$), we find that the only time the "$-$" sign actually appears is for odd m and links of 2 components. That this is true for case $m = 1$ (which is in fact the Alexander–Conway polynomial) is well-known [16]. These results are exemplified in our previous work [4, 6] for LG^2.

Lastly, we often wish to eliminate α from expressions of the form $q^{x\alpha + y}$, to express them in terms of p and q alone. Using (15), we have:

$$q^{x\alpha + y} = p^x q^{y - x(\frac{m-n}{2})}.$$

3.5 $LG^{m,n}$ of split links

Recall that we define $LG_K^{m,n}$ as a $(1,1)$ tangle invariant, obtained for a link K as the first component of the diagonal tensor (scalar multiple of the identity) T_K. We do this as the closed form (i.e. the $(0,0)$ tangle form) always evaluates to zero (cf. the ADO invariant [1]).

To see this, begin by observing that the value of our state model on 0_1 (i.e. the unknot, an isolated loop) as a $(0,0)$ tangle is zero, as $\sum_a C_a^a = 0$. This follows from the fact that for the $U_q[gl(m|n)]$ superalgebras, the q-superdimension of typical representations (defined by $\mathrm{str}[\pi(q^{2h_\rho})]$) is always identically zero [18]. As S is a grading-stripped version of the exponential of the Cartan element $\pi(q^{2h_\rho})$, we necessarily have $\mathrm{tr}(S) = 0$, hence $\mathrm{tr}(C) = 0$. Multiplying these results by the scalar in T_K yields the result.

Now, let $K = K_1 \sqcup K_2$ be the split (i.e. disconnected, separated) union of links K_1 and K_2, and say that we are trying to evaluate the $(1,1)$ tangle form using a string of K_1. The construction of $LG_K^{m,n}$ means that at some point of contracting Z_K to T_K, we close the final string of K_2, and at this stage our tensor becomes zero throughout, thus T_K is zero. Thus, as disconnected multicomponent links represented by $(1,1)$ tangles necessarily include a closed component, we have proven:

Theorem 2 $LG_K^{m,n} = 0$ *for disconnected multicomponent links* K.

4 Computational issues in evaluating $LG^{m,n}$

4.1 Various sets of computational variables

The representation of the braid generator σ obtained from the representation theory [3, 5] contains algebraic expressions in variables q and α, including many q brackets. This form is readable to human eyes, but can be improved upon for machine consumption. We shall call $\{q, \alpha\}$ the **rep(resentation)** variables.

From (15), we see that our link invariants are naturally expressed in terms of q and $p \triangleq q^{\alpha + \frac{m-n}{2}}$; so we initially make this change of variables in the internal representation of the braid generator and the positive handle. This action replaces all the q brackets, which contained α. The resulting braid generator contains rational expressions in variables $q^{\frac{1}{2}}$ and p. To simplify the vulgar fractions within the exponents, we define a new variable to be used internally: $Q \triangleq q^{\frac{1}{2}}$. In some sense, the resulting braid generator is now optimally literate, and we use this form to accrete tensors to build Z_K, and also to check the QYBE and the Matveev Δ–∇ test. We shall call $\{Q, p\}$ the **int(ernal)** variables, and to convert from **rep** to **int** variables, we shall invoke *in order* the following rules:

$$\left\{ q^{x\alpha+y} \mapsto p^x q^{y - x(\frac{m-n}{2})}, \qquad q \mapsto Q^2 \right\},$$

where $x, y \in \mathbb{Z}$. We occasionally have an interest in the inverse transformation to convert from **int** to **rep** variables, and for this we shall invoke *in order* the following rules:

$$\left\{ Q \mapsto q^{\frac{1}{2}}, \qquad p \mapsto q^{\alpha + \frac{m-n}{2}}, \qquad (q^{x\alpha+y} - \overline{q}^{x\alpha+y}) \mapsto (q - \overline{q})[x\alpha + y]_q \right\},$$

where handling the last of these rules typically requires some care.

Sometimes, we must invert the **int** variables, for example in computing the inverse braid generator $\overline{\sigma}$. We have the rules:

$$\left\{Q \mapsto \overline{Q}, \qquad p \mapsto \overline{p}\right\}.$$

Finally, extracting the first component of T_K thus yields an expression **int** variables. We must then expand $Q \mapsto q^2$. Furthermore, we discover that $LG_K^{m,n}$ is actually an invariant in p^2 not just p, so we define $P = p^2$ to reduce things a little. We shall call $\{q, P\}$ the **L(ink) I(nvariant)** variables, and to convert from **int** to **LI** variables, we shall invoke the following rules:

$$\left\{Q \mapsto q^{\frac{1}{2}}, \qquad p \mapsto P^{\frac{1}{2}}\right\}.$$

Parameters used for the state models for LG^m for $m = 1, 2, 3, 4$ are presented in Appendix A.

4.2 Explicit construction of S

From (14), we have for $U_q[gl(m|n)]$ that $S \equiv \pi(q^{2h_\rho})$ is:

$$\begin{aligned} S \;=\; & \pi(K_1)^{2(m-1)}\pi(K_2)^{2(m-2)} \cdots \pi(K_{m-1})^2 \cdot \\ & \pi(K_{m+2})^{-2}\pi(K_{m+3})^{-4} \cdots \pi(K_{m+n})^{-2(n-1)}. \end{aligned} \tag{17}$$

Setting $n = 1$ in (17), we have:

$$S = \pi_\Lambda(K_1)^{2(m-1)}\pi_\Lambda(K_2)^{2(m-2)} \cdots \pi_\Lambda(K_{m-1})^2.$$

To illustrate, for the $U_q[gl(2|1)]$ case, we have $h_\rho = E^1{}_1$, hence $q^{h_\rho} = K_1$, so $S = \pi(K_1)^2$. This contrasts with the choice of $\theta = -1$ made in [2], which yields $h_\rho = -E^2{}_2 - E^3{}_3$, viz $q^{h_\rho} = q^{-E^2{}_2}q^{-E^3{}_3} = K_2^{-1}K_3$, so $S = \pi(K_2)^{-2}\pi(K_3)^2$.

4.3 Illustrative examples of LG^m

At present, we are able to compute state model parameters for $LG^{m,1} \equiv LG^m$ only, as we have not yet computed $\check{R}$ or matrix elements for the K_i for cases $n \neq 1$. For the cases $m = 1, 2, 3, 4$, we are able to make the following comments.

- LG^1 is the Alexander–Conway polynomial in variable $P \equiv q^{2\alpha}$. This is a well-known result, cf. [15, 24].
- Evaluations for LG^2 for all prime knots of up to 10 crossings have been reported in [4]. In that paper, we claimed that LG^m for $m > 2$ was essentially incomputable due to vast memory requirements of the tensors Z_K; but we have since made some headway in this by adapting our code to recognise the sparsity of these tensors; doing the symbolic equivalent of what is called "sparse matrix multiplication" in numerical linear algebra. This change comes at a cost of more lines of interpreted code, but is still an improvement in algorithmic efficiency. It also results in an increase in the speed of computation for LG^2, and facilitates its evaluation from braid presentations of 6 strings, something not previously feasible.

- Evaluations for LG^3 and LG^4 for various links are presented in Appendix B. Those lists are quite brief, and only include some links of braid index at most 3. Our current computational method requires too much memory for us to extend our tables of polynomials any further.

Of some interest is the rate of growth in exponent of the polynomials with m for a particular link. For example, we have the following results for the trefoil knot 3_1 and the figure eight knot 4_1:

$$
\begin{aligned}
LG^1_{3_1} &= -(1) \\
&\quad +(\overline{P}^1 + P^1)(+1)
\end{aligned}
$$

$$
\begin{aligned}
LG^2_{3_1} &= (1+2q^2) \\
&\quad -(\overline{P}^1 + P^1)(q+q^3) \\
&\quad +(\overline{P}^2 + P^2)(q^2)
\end{aligned}
$$

$$
\begin{aligned}
LG^3_{3_1} &= -(q^2+2q^4+3q^6+q^8) \\
&\quad +(\overline{P}^1 + P^1)(q^2+2q^4+2q^6+q^8) \\
&\quad -(\overline{P}^2 + P^2)(q^4+q^6+q^8) \\
&\quad +(\overline{P}^3 + P^3)(q^6)
\end{aligned}
$$

$$
\begin{aligned}
LG^4_{3_1} &= (q^4+2q^6+4q^8+4q^{10}+5q^{12}+2q^{14}+q^{16}) \\
&\quad -(\overline{P}^1 + P^1)(q^5+2q^7+4q^9+4q^{11}+3q^{13}+2q^{15}) \\
&\quad +(\overline{P}^2 + P^2)(q^6+2q^8+2q^{10}+3q^{12}+q^{14}+q^{16}) \\
&\quad -(\overline{P}^3 + P^3)(q^9+q^{11}+q^{13}+q^{15}) \\
&\quad +(\overline{P}^4 + P^4)(q^{12})
\end{aligned}
$$

$$
\begin{aligned}
LG^1_{4_1} &= (3) \\
&\quad -(\overline{P}^1 + P^1)(1)
\end{aligned}
$$

$$
\begin{aligned}
LG^2_{4_1} &= (2\overline{q}^2+7+2q^2) \\
&\quad -(\overline{P}^1 + P^1)(3\overline{q}+3q) \\
&\quad +(\overline{P}^2 + P^2)(1)
\end{aligned}
$$

$$
\begin{aligned}
LG^3_{4_1} &= (5\overline{q}^4+9\overline{q}^2+17+9q^2+5q^4) \\
&\quad -(\overline{P}^1 + P^1)(2\overline{q}^4+8\overline{q}^2+10+8q^2+2q^4) \\
&\quad +(\overline{P}^2 + P^2)(3\overline{q}^2+3+3q^2) \\
&\quad -(\overline{P}^3 + P^3)(1)
\end{aligned}
$$

5 Further work

The current work is part of a larger program to automate the construction of more general quantum link invariants. A few comments on the direction of this program are in order.

- In this paper, the limits of our method of evaluation have been reached,[6] and a more efficient method of evaluation is required. A promising candidate involves chasing through braids one crossing at a time, accumulating only an $N \times N$ matrix (where $N = \dim(\mathrm{V})$) of polynomials at each step. That method requires foreknowledge of the decomposition of $\check{R}$ into the canonical form $\check{R} = \sum_i a_i \otimes b_i$, and this is already available for $U_q[sl(2)]$ and $U_q[gl(1|1)]$. It is applicable to links of any number of crossings and components, and is really only limited by N, although much less strongly than our current method. In particular, it is not dependent on the string index of braid presentations.
- Moreover, the construction of more general quantum link invariants requires a more general approach to construction of underlying R matrices. The current method [3, 5] exploits explicit knowledge of the decomposition of the tensor product of the underlying module, but this is *not* generally known. Alternatively, it is also possible to construct explicit R matrices from knowledge of the universal (i.e. representation-independent) R matrix and the matrix elements of the underlying representation. As we have to hand details of the universal R matrices for arbitrary quantum (super)algebras [13] (albeit in a somewhat abstract form), and some knowledge of a process to construct the matrix elements, it is eminently possible to construct many more R matrices.
- Lastly, we are limited by our use of braids, for which we have systematic tables only for the first 249 prime knots of up to 10 crossings. As of 1998, Dowker codes for all the $1,701,936$ prime knots of up to 16 crossings have been enumerated [9], and our not being able to access them is a sad thing. As we don't have the implementation of an algorithm that allows us to map these Dowker codes to braids, it is attractive to try to adapt new material to accept Dowker codes as input. The converse to this is that our new invariants $LG^{m,n}$ are well suited to extending those tables, as they distinguish many more knots than other polynomial invariants.

Acknowledgements

I am grateful to Jon Links of The University of Queensland, Australia, for advice on how the deduce the natural choice of variables to describe $LG^{m,n}$, and other worthwhile discussions.

My research at Kyoto University is funded by a Postdoctoral Fellowship for Foreign Researchers (# P99703), provided by the Japan Society for the Promotion of Science. Dōmo arigatō gozaimashita!

[6] The material has also been applied to the evaluation of 'N-Jones' polynomials V^N. These are the quantum link invariants associated with the N dimensional representations of $U_q[sl(2)]$. In the language of [14], they are monochromatic versions of coloured Jones polynomials of order N. The limit to computation for these invariants for prime knots of up to 10 crossings is around $N = 4$, although we can calculate $V_{3_1}^{13}$.

A State model parameters

Below, we list state model parameters for LG^m, for $m = 1, 2, 3, 4$. To improve literacy, we have written $[X]$ for $[X]_q$, $\overline{X}$ for X^{-1}, for various X, and $\Delta = q - \overline{q}$. Horizontal lines divide tensor components into symmetry classes.

Parameters for LG^1

The braid generator σ has 5 nonzero components:

$$q^{-\alpha}\left\{e^{11}_{11}\right\}, \qquad -q^{\alpha}\left\{e^{22}_{22}\right\}, \qquad -\Delta[\alpha]\left\{e^{21}_{21}\right\}, \qquad 1\begin{Bmatrix} e^{12}_{21} \\ e^{21}_{12} \end{Bmatrix},$$

and the left handle C has 2 diagonal components:

$$C = q^{-\alpha}\begin{Bmatrix} +e^{1}_{1} \\ -e^{2}_{2} \end{Bmatrix},$$

using the scaling factors:

$$\kappa_\sigma = q^{-\alpha}, \qquad \kappa_C = q^{-\alpha}.$$

Parameters for LG^2

The braid generator σ has 26 nonzero components:

$$q^{-2\alpha}\left\{e^{11}_{11}\right\}, \qquad -1\left\{e^{22}_{22}, e^{33}_{33}\right\}, \qquad q^{2\alpha+2}\left\{e^{44}_{44}\right\},$$

$$-\Delta q^{-\alpha}[\alpha]\left\{e^{21}_{21}, e^{31}_{31}\right\}, \qquad \Delta^2 q[\alpha][\alpha+1]\left\{e^{41}_{41}\right\},$$
$$\Delta q^{\alpha+1}[\alpha+1]\left\{e^{42}_{42}, e^{43}_{43}\right\}, \qquad \Delta q\left\{e^{32}_{32}\right\},$$

$$1\begin{Bmatrix} e^{14}_{41} \\ e^{41}_{14} \end{Bmatrix}, \qquad -q\begin{Bmatrix} e^{23}_{32} \\ e^{32}_{23} \end{Bmatrix}, \qquad q^{-\alpha}\begin{Bmatrix} e^{12}_{21}, e^{13}_{31} \\ e^{21}_{12}, e^{31}_{13} \end{Bmatrix}, \qquad q^{\alpha+1}\begin{Bmatrix} e^{24}_{42}, e^{34}_{43} \\ e^{42}_{24}, e^{43}_{34} \end{Bmatrix},$$

$$\Delta q[\alpha]^{\frac{1}{2}}[\alpha+1]^{\frac{1}{2}}\left\{-\overline{q}^{\frac{1}{2}}\begin{Bmatrix} e^{23}_{41} \\ e^{41}_{23} \end{Bmatrix}, +q^{\frac{1}{2}}\begin{Bmatrix} e^{32}_{41} \\ e^{41}_{32} \end{Bmatrix}\right\},$$

and the left handle C has 4 diagonal components:

$$C = q^{-2\alpha-1}\left\{\begin{array}{l} +q\left\{e^{1}_{1}\right\} \\ -\left\{qe^{2}_{2},\ \overline{q}e^{3}_{3}\right\} \\ +\overline{q}\left\{e^{4}_{4}\right\} \end{array}\right\},$$

using the scaling factors:

$$\kappa_\sigma = q^{-2\alpha}, \qquad \kappa_C = q^{-2\alpha}.$$

Parameters for LG^3

The braid generator σ has 139 nonzero components:

$$q^{-3\alpha}\left\{e^{11}_{11}\right\}, \quad -q^{-\alpha}\left\{e^{22}_{22}, e^{33}_{33}, e^{44}_{44}\right\}, \quad q^{\alpha+2}\left\{e^{55}_{55}, e^{66}_{66}, e^{77}_{77}\right\}, \quad -q^{3\alpha+6}\left\{e^{88}_{88}\right\},$$

$$-\Delta q^{-2\alpha}[\alpha]\left\{e^{21}_{21}, e^{31}_{31}, e^{41}_{41}\right\}, \qquad \Delta^2 q^{-\alpha+1}[\alpha][\alpha+1]\left\{e^{51}_{51}, e^{61}_{61}, e^{71}_{71}\right\},$$
$$-\Delta q^{2\alpha+4}[\alpha+2]\left\{e^{87}_{87}, e^{86}_{86}, e^{85}_{85}\right\}, \qquad -\Delta^2 q^{\alpha+3}[\alpha+1][\alpha+2]\left\{e^{84}_{84}, e^{83}_{83}, e^{82}_{82}\right\},$$
$$\Delta q[\alpha+1]\left\{e^{52}_{52}, e^{53}_{53}, e^{62}_{62}, e^{64}_{64}, e^{73}_{73}, e^{74}_{74}\right\},$$
$$\Delta q^{-\alpha+1}\left\{e^{32}_{32}, e^{42}_{42}, e^{43}_{43}\right\}, \qquad -\Delta q^{\alpha+3}\left\{e^{65}_{65}, e^{75}_{75}, e^{76}_{76}\right\},$$
$$\Delta q^2[\alpha+1]\left\{-\Delta e^{63}_{63}, q(\overline{q}^2-q^2)e^{72}_{72}\right\}, \qquad -\Delta^3 q^3[\alpha][\alpha+1][\alpha+2]\left\{e^{81}_{81}\right\},$$

$$q^{-2\alpha}\left\{\begin{matrix} e^{12}_{21}, e^{13}_{31}, e^{14}_{41} \\ e^{21}_{12}, e^{31}_{13}, e^{41}_{14} \end{matrix}\right\}, \qquad q^{-\alpha}\left\{\begin{matrix} e^{15}_{51}, e^{16}_{61}, e^{17}_{71} \\ e^{51}_{15}, e^{61}_{16}, e^{71}_{17} \end{matrix}\right\},$$
$$q^{2\alpha+4}\left\{\begin{matrix} e^{85}_{58}, e^{86}_{68}, e^{87}_{78} \\ e^{58}_{85}, e^{68}_{86}, e^{78}_{87} \end{matrix}\right\}, \qquad -q^{\alpha+2}\left\{\begin{matrix} e^{82}_{28}, e^{83}_{38}, e^{84}_{48} \\ e^{28}_{82}, e^{38}_{83}, e^{48}_{84} \end{matrix}\right\},$$
$$-q^{-\alpha+1}\left\{\begin{matrix} e^{23}_{32}, e^{24}_{42}, e^{34}_{43} \\ e^{32}_{23}, e^{42}_{24}, e^{43}_{34} \end{matrix}\right\}, \qquad q^{\alpha+3}\left\{\begin{matrix} e^{56}_{65}, e^{57}_{75}, e^{67}_{76} \\ e^{65}_{56}, e^{75}_{57}, e^{76}_{67} \end{matrix}\right\},$$
$$q\left\{\begin{matrix} e^{25}_{52}, e^{26}_{62}, e^{35}_{53}, e^{37}_{73}, e^{46}_{64}, e^{47}_{74} \\ e^{52}_{25}, e^{62}_{26}, e^{53}_{35}, e^{73}_{37}, e^{64}_{46}, e^{74}_{47} \end{matrix}\right\}, \qquad q^2\left\{\begin{matrix} e^{27}_{72}, e^{45}_{54}, e^{36}_{63} \\ e^{72}_{27}, e^{54}_{45}, e^{63}_{36} \end{matrix}\right\}, \qquad 1\left\{\begin{matrix} e^{18}_{81} \\ e^{81}_{18} \end{matrix}\right\},$$

$$-\Delta q^2\left\{+1\left\{\begin{matrix} e^{45}_{63} \\ e^{63}_{45} \end{matrix}\right\}, -q\left\{\begin{matrix} e^{45}_{72} \\ e^{72}_{45} \end{matrix}\right\}, +1\left\{\begin{matrix} e^{36}_{72} \\ e^{72}_{36} \end{matrix}\right\}\right\},$$
$$\Delta q^3[\alpha+1]\left\{+\overline{q}\left\{\begin{matrix} e^{54}_{63} \\ e^{63}_{54} \end{matrix}\right\}, -1\left\{\begin{matrix} e^{54}_{72} \\ e^{72}_{54} \end{matrix}\right\}, +q\left\{\begin{matrix} e^{63}_{72} \\ e^{72}_{63} \end{matrix}\right\}\right\},$$
$$\Delta q^2[\alpha]^{\frac{1}{2}}[\alpha+2]^{\frac{1}{2}}\left\{-\overline{q}\left\{\begin{matrix} e^{27}_{81} \\ e^{81}_{27} \end{matrix}\right\}, +1\left\{\begin{matrix} e^{36}_{81} \\ e^{81}_{36} \end{matrix}\right\}, -q\left\{\begin{matrix} e^{45}_{81} \\ e^{81}_{45} \end{matrix}\right\}\right\},$$
$$\Delta q^{-\alpha+1}[\alpha]^{\frac{1}{2}}[\alpha+1]^{\frac{1}{2}}\left\{-\overline{q}^{\frac{1}{2}}\left\{\begin{matrix} e^{23}_{51}, e^{24}_{61}, e^{34}_{71} \\ e^{51}_{23}, e^{61}_{24}, e^{71}_{34} \end{matrix}\right\}, +q^{\frac{1}{2}}\left\{\begin{matrix} e^{32}_{51}, e^{42}_{61}, e^{43}_{71} \\ e^{51}_{32}, e^{61}_{42}, e^{71}_{43} \end{matrix}\right\}\right\},$$
$$\Delta q^{\alpha+3}[\alpha+1]^{\frac{1}{2}}[\alpha+2]^{\frac{1}{2}}\left\{-\overline{q}^{\frac{1}{2}}\left\{\begin{matrix} e^{56}_{82}, e^{57}_{83}, e^{67}_{84} \\ e^{82}_{56}, e^{83}_{57}, e^{84}_{67} \end{matrix}\right\}, +q^{\frac{1}{2}}\left\{\begin{matrix} e^{65}_{82}, e^{75}_{83}, e^{76}_{84} \\ e^{82}_{65}, e^{83}_{75}, e^{84}_{76} \end{matrix}\right\}\right\},$$
$$\Delta^2 q^3[\alpha]^{\frac{1}{2}}[\alpha+1][\alpha+2]^{\frac{1}{2}}\left\{+\overline{q}\left\{\begin{matrix} e^{54}_{81} \\ e^{81}_{54} \end{matrix}\right\}, -1\left\{\begin{matrix} e^{63}_{81} \\ e^{81}_{63} \end{matrix}\right\}, +q\left\{\begin{matrix} e^{72}_{81} \\ e^{81}_{72} \end{matrix}\right\}\right\},$$

and the left handle C has 8 diagonal components:

$$C = q^{-3\alpha-3}\left\{\begin{array}{l} +q^3\left\{e^1_1\right\} \\ -q\left\{q^2 e^2_2,\ \ e^3_3,\ \ \overline{q}^2 e^4_4\right\} \\ +\overline{q}\left\{\overline{q}^2 e^7_7,\ \ e^6_6,\ \ q^2 e^5_5\right\} \\ -\overline{q}^3\left\{e^8_8\right\} \end{array}\right\},$$

using the scaling factors:

$$\kappa_\sigma = q^{-3\alpha}, \qquad \kappa_C = q^{-3\alpha}.$$

Parameters for LG^4

The reader will have by now appreciated the recurring patterns in the components of our R matrices. To save space, we introduce a little more notation, which eliminates the q brackets altogether. To whit, we write:

$$A_i^z \quad \triangleq \quad [\alpha+i]_q{}^z, \qquad \text{where } z \in \{\tfrac{1}{2}, 1\},$$

and $i \in \{0,1,2,3\}$. With this notation, the braid generator σ has 758 nonzero components:

$$q^{-4\alpha}\left\{e_{1,1}^{1,1}\right\}, \qquad q^{2}\left\{e_{6,6}^{6,6}, e_{7,7}^{7,7}, e_{8,8}^{8,8}, e_{9,9}^{9,9}, e_{10,10}^{10,10}, e_{11,11}^{11,11}\right\}, \qquad q^{4\alpha+12}\left\{e_{16,16}^{16,16}\right\},$$

$$-q^{-2\alpha}\left\{e_{2,2}^{2,2}, e_{3,3}^{3,3}, e_{4,4}^{4,4}, e_{5,5}^{5,5}\right\}, \qquad -q^{2\alpha+6}\left\{e_{15,15}^{15,15}, e_{14,14}^{14,14}, e_{13,13}^{13,13}, e_{12,12}^{12,12}\right\},$$

$$-\Delta q^{-3\alpha}A_0\left\{e_{2,1}^{2,1}, e_{3,1}^{3,1}, e_{4,1}^{4,1}, e_{5,1}^{5,1}\right\}, \quad \Delta q^{3\alpha+9}A_3\left\{e_{16,12}^{16,12}, e_{16,13}^{16,13}, e_{16,14}^{16,14}, e_{16,15}^{16,15}\right\},$$

$$-\Delta q^{3}\left\{e_{7,6}^{7,6}, e_{8,6}^{8,6}, e_{8,7}^{8,7}, e_{9,6}^{9,6}, e_{9,7}^{9,7}, e_{10,6}^{10,6}, e_{10,8}^{10,8}, e_{10,9}^{10,9}, e_{11,7}^{11,7}, e_{11,8}^{11,8}, e_{11,9}^{11,9}, e_{11,10}^{11,10}\right\},$$

$$\Delta q^{-2\alpha+1}\left\{e_{3,2}^{3,2}, e_{4,2}^{4,2}, e_{4,3}^{4,3}, e_{5,2}^{5,2}, e_{5,3}^{5,3}, e_{5,4}^{5,4}\right\},$$

$$\Delta q^{2\alpha+7}\left\{e_{13,12}^{13,12}, e_{14,12}^{14,12}, e_{14,13}^{14,13}, e_{15,12}^{15,12}, e_{15,13}^{15,13}, e_{15,14}^{15,14}\right\},$$

$$\Delta q^{-\alpha+1}A_1\left\{e_{6,2}^{6,2}, e_{6,3}^{6,3}, e_{7,2}^{7,2}, e_{7,4}^{7,4}, e_{8,2}^{8,2}, e_{8,5}^{8,5}, e_{9,3}^{9,3}, e_{9,4}^{9,4}, e_{10,3}^{10,3}, e_{10,5}^{10,5}, e_{11,4}^{11,4}, e_{11,5}^{11,5}\right\},$$

$$\Delta q^{-\alpha+3}(\overline{q}^2-q^2)A_1\left\{e_{9,2}^{9,2}, e_{10,2}^{10,2}, e_{11,2}^{11,2}, e_{11,3}^{11,3}\right\},$$

$$\Delta^2 q^{-2\alpha+1}A_0A_1\left\{e_{6,1}^{6,1}, e_{7,1}^{7,1}, e_{8,1}^{8,1}, e_{9,1}^{9,1}, e_{10,1}^{10,1}, e_{11,1}^{11,1}\right\},$$

$$-\Delta q^{\alpha+4}A_2\left\{\begin{matrix} e_{12,6}^{12,6}, e_{12,7}^{12,7}, e_{12,9}^{12,9}, e_{13,6}^{13,6}, e_{13,8}^{13,8}, e_{13,10}^{13,10} \\ e_{14,7}^{14,7}, e_{14,8}^{14,8}, e_{14,11}^{14,11}, e_{15,9}^{15,9}, e_{15,10}^{15,10}, e_{15,11}^{15,11} \end{matrix}\right\},$$

$$-\Delta^2 q^{-\alpha+2}A_1\left\{e_{7,3}^{7,3}, e_{8,3}^{8,3}, e_{8,4}^{8,4}, e_{10,4}^{10,4}\right\}, \qquad \Delta^2 q^{\alpha+5}A_2\left\{e_{13,7}^{13,7}, e_{13,9}^{13,9}, e_{14,9}^{14,9}, e_{14,10}^{14,10}\right\},$$

$$-\Delta q^{\alpha+6}(\overline{q}^2-q^2)A_2\left\{e_{14,6}^{14,6}, e_{15,6}^{15,6}, e_{15,7}^{15,7}, e_{15,8}^{15,8}\right\},$$

$$-\Delta^2 q^{3}A_1A_2\left\{\begin{matrix} e_{12,2}^{12,2}, e_{12,3}^{12,3}, e_{12,4}^{12,4}, e_{13,2}^{13,2}, e_{13,3}^{13,3}, e_{13,5}^{13,5} \\ e_{14,2}^{14,2}, e_{14,4}^{14,4}, e_{14,5}^{14,5}, e_{15,3}^{15,3}, e_{15,4}^{15,4}, e_{15,5}^{15,5} \end{matrix}\right\},$$

$$-\Delta^3 q^{-\alpha+3}A_0A_1A_2\left\{e_{12,1}^{12,1}, e_{13,1}^{13,1}, e_{14,1}^{14,1}, e_{15,1}^{15,1}\right\},$$

$$\Delta^2 q^{2\alpha+7}A_2A_3\left\{e_{16,6}^{16,6}, e_{16,7}^{16,7}, e_{16,8}^{16,8}, e_{16,9}^{16,9}, e_{16,10}^{16,10}, e_{16,11}^{16,11}\right\},$$

$$\Delta^3 q^{\alpha+6}A_1A_2A_3\left\{e_{16,2}^{16,2}, e_{16,3}^{16,3}, e_{16,4}^{16,4}, e_{16,5}^{16,5}\right\}, \qquad \Delta^3 q^{6}(\overline{q}^2+1+q^2)A_1A_2\left\{e_{15,2}^{15,2}\right\},$$

$$\Delta^2 q^{4}\left\{e_{10,7}^{10,7}\right\}, \qquad \Delta^2 q^{5}(q+\overline{q})\left\{e_{11,6}^{11,6}\right\}, \qquad \Delta^3 q^{4}A_1A_2\left\{e_{13,4}^{13,4}\right\},$$

$$-\Delta^2 q^{5}(\overline{q}^2-q^2)A_1A_2\left\{e_{14,3}^{14,3}\right\}, \quad \Delta^4 q^{6}A_0A_1A_2A_3\left\{e_{16,1}^{16,1}\right\},$$

$$q^{-3\alpha}\left\{\begin{matrix} e_{2,1}^{1,2}, e_{3,1}^{1,3}, e_{4,1}^{1,4}, e_{5,1}^{1,5} \\ e_{1,2}^{2,1}, e_{1,3}^{3,1}, e_{1,4}^{4,1}, e_{1,5}^{5,1} \end{matrix}\right\}, \qquad q^{2\alpha+6}\left\{\begin{matrix} e_{16,11}^{11,16}, e_{16,10}^{10,16}, e_{16,9}^{9,16}, e_{16,8}^{8,16}, e_{16,7}^{7,16}, e_{16,6}^{6,16} \\ e_{11,16}^{16,11}, e_{10,16}^{16,10}, e_{9,16}^{16,9}, e_{8,16}^{16,8}, e_{7,16}^{16,7}, e_{6,16}^{16,6} \end{matrix}\right\},$$

$$q^{3\alpha+9}\left\{\begin{matrix} e^{16,15}_{15,16}, e^{16,14}_{14,16}, e^{16,13}_{13,16}, e^{16,12}_{12,16} \\ e^{15,16}_{16,15}, e^{14,16}_{16,14}, e^{13,16}_{16,13}, e^{12,16}_{16,12} \end{matrix}\right\}, \qquad q^{-2\alpha}\left\{\begin{matrix} e^{1,6}_{6,1}, e^{1,7}_{7,1}, e^{1,8}_{8,1}, e^{1,9}_{9,1}, e^{1,10}_{10,1}, e^{1,11}_{11,1} \\ e^{6,1}_{1,6}, e^{7,1}_{1,7}, e^{8,1}_{1,8}, e^{9,1}_{1,9}, e^{10,1}_{1,10}, e^{11,1}_{1,11} \end{matrix}\right\},$$

$$q^{-\alpha}\left\{\begin{matrix} e^{1,12}_{12,1}, e^{1,13}_{13,1}, e^{1,14}_{14,1}, e^{1,15}_{15,1} \\ e^{12,1}_{1,12}, e^{13,1}_{1,13}, e^{14,1}_{1,14}, e^{15,1}_{1,15} \end{matrix}\right\}, \qquad -q^{-2\alpha+1}\left\{\begin{matrix} e^{2,3}_{3,2}, e^{2,4}_{4,2}, e^{2,5}_{5,2}, e^{3,4}_{4,3}, e^{3,5}_{5,3}, e^{4,5}_{5,4} \\ e^{3,2}_{2,3}, e^{4,2}_{2,4}, e^{5,2}_{2,5}, e^{4,3}_{3,4}, e^{5,3}_{3,5}, e^{5,4}_{4,5} \end{matrix}\right\},$$

$$q^{\alpha+3}\left\{\begin{matrix} e^{5,16}_{16,5}, e^{4,16}_{16,4}, e^{3,16}_{16,3}, e^{2,16}_{16,2} \\ e^{16,5}_{5,16}, e^{16,4}_{4,16}, e^{16,3}_{3,16}, e^{16,2}_{2,16} \end{matrix}\right\}, \quad -q^{2\alpha+7}\left\{\begin{matrix} e^{15,14}_{14,15}, e^{15,13}_{13,15}, e^{15,12}_{12,15}, e^{14,13}_{13,14}, e^{14,12}_{12,14}, e^{13,12}_{12,13} \\ e^{14,15}_{15,14}, e^{13,15}_{15,13}, e^{12,15}_{15,12}, e^{13,14}_{14,13}, e^{12,14}_{14,12}, e^{12,13}_{13,12} \end{matrix}\right\},$$

$$q^{-\alpha+1}\left\{\begin{matrix} e^{2,6}_{6,2}, e^{2,7}_{7,2}, e^{2,8}_{8,2}, e^{3,6}_{6,3}, e^{3,9}_{9,3}, e^{3,10}_{10,3}, e^{4,7}_{7,4}, e^{4,9}_{9,4}, e^{4,11}_{11,4}, e^{5,8}_{8,5}, e^{5,10}_{10,5}, e^{5,11}_{11,5} \\ e^{6,2}_{2,6}, e^{7,2}_{2,7}, e^{8,2}_{2,8}, e^{6,3}_{3,6}, e^{9,3}_{3,9}, e^{10,3}_{3,10}, e^{7,4}_{4,7}, e^{9,4}_{4,9}, e^{11,4}_{4,11}, e^{8,5}_{5,8}, e^{10,5}_{5,10}, e^{11,5}_{5,11} \end{matrix}\right\},$$

$$q^{\alpha+4}\left\{\begin{matrix} e^{15,10}_{10,15}, e^{15,11}_{11,15}, e^{15,9}_{9,15}, e^{14,11}_{11,14}, e^{14,8}_{8,14}, e^{14,7}_{7,14}, e^{13,10}_{10,13}, e^{13,8}_{8,13}, e^{13,6}_{6,13}, e^{12,9}_{9,12}, e^{12,7}_{7,12}, e^{12,6}_{6,12} \\ e^{11,15}_{15,11}, e^{10,15}_{15,10}, e^{9,15}_{15,9}, e^{11,14}_{14,11}, e^{8,14}_{14,8}, e^{7,14}_{14,7}, e^{10,13}_{13,10}, e^{8,13}_{13,8}, e^{6,13}_{13,6}, e^{9,12}_{12,9}, e^{7,12}_{12,7}, e^{6,12}_{12,6} \end{matrix}\right\},$$

$$q^{-\alpha+2}\left\{\begin{matrix} e^{2,9}_{9,2}, e^{2,10}_{10,2}, e^{2,11}_{11,2}, e^{3,7}_{7,3}, e^{3,8}_{8,3}, e^{3,11}_{11,3}, e^{4,6}_{6,4}, e^{4,8}_{8,4}, e^{4,10}_{10,4}, e^{5,6}_{6,5}, e^{5,7}_{7,5}, e^{5,9}_{9,5} \\ e^{9,2}_{2,9}, e^{10,2}_{2,10}, e^{11,2}_{2,11}, e^{7,3}_{3,7}, e^{8,3}_{3,8}, e^{11,3}_{3,11}, e^{6,4}_{4,6}, e^{8,4}_{4,8}, e^{10,4}_{4,10}, e^{6,5}_{5,6}, e^{7,5}_{5,7}, e^{9,5}_{5,9} \end{matrix}\right\},$$

$$q^{3}\left\{\begin{matrix} e^{6,7}_{7,6}, e^{6,8}_{8,6}, e^{6,9}_{9,6}, e^{6,10}_{10,6}, e^{7,8}_{8,7}, e^{7,9}_{9,7}, e^{7,11}_{11,7}, e^{8,10}_{10,8}, e^{8,11}_{11,8}, e^{9,10}_{10,9}, e^{9,11}_{11,9}, e^{10,11}_{11,10} \\ e^{7,6}_{6,7}, e^{8,6}_{6,8}, e^{9,6}_{6,9}, e^{10,6}_{6,10}, e^{8,7}_{7,8}, e^{9,7}_{7,9}, e^{10,8}_{8,10}, e^{11,7}_{7,11}, e^{11,8}_{8,11}, e^{10,9}_{9,10}, e^{11,9}_{9,11}, e^{11,10}_{10,11} \end{matrix}\right\},$$

$$1\left\{\begin{matrix} e^{1,16}_{16,1} \\ e^{16,1}_{1,16} \end{matrix}\right\}, \qquad -q^{3}\left\{\begin{matrix} e^{2,15}_{15,2}, e^{3,14}_{14,3}, e^{4,13}_{13,4}, e^{5,12}_{12,5} \\ e^{12,5}_{5,12}, e^{13,4}_{4,13}, e^{14,3}_{3,14}, e^{15,2}_{2,15} \end{matrix}\right\}, \qquad q^{4}\left\{\begin{matrix} e^{6,11}_{11,6}, e^{7,10}_{10,7}, e^{8,9}_{9,8} \\ e^{11,6}_{6,11}, e^{10,7}_{7,10}, e^{9,8}_{8,9} \end{matrix}\right\},$$

$$q^{\alpha+5}\left\{\begin{matrix} e^{6,14}_{14,6}, e^{6,15}_{15,6}, e^{7,13}_{13,7}, e^{7,15}_{15,7}, e^{8,12}_{12,8}, e^{8,15}_{15,8}, e^{9,13}_{13,9}, e^{9,14}_{14,9}, e^{10,12}_{12,10}, e^{10,14}_{14,10}, e^{11,12}_{12,11}, e^{11,13}_{13,11} \\ e^{14,6}_{6,14}, e^{15,6}_{6,15}, e^{13,7}_{7,13}, e^{15,7}_{7,15}, e^{12,8}_{8,12}, e^{15,8}_{8,15}, e^{13,9}_{9,13}, e^{14,9}_{9,14}, e^{12,10}_{10,12}, e^{14,10}_{10,14}, e^{12,11}_{11,12}, e^{13,11}_{11,13} \end{matrix}\right\},$$

$$-q^{2}\left\{\begin{matrix} e^{2,12}_{12,2}, e^{2,13}_{13,2}, e^{2,14}_{14,2}, e^{3,12}_{12,3}, e^{3,13}_{13,3}, e^{3,15}_{15,3}, e^{4,12}_{12,4}, e^{4,14}_{14,4}, e^{4,15}_{15,4}, e^{5,13}_{13,5}, e^{5,14}_{14,5}, e^{5,15}_{15,5} \\ e^{12,2}_{2,12}, e^{12,3}_{3,12}, e^{12,4}_{4,12}, e^{13,2}_{2,13}, e^{13,3}_{3,13}, e^{13,5}_{5,13}, e^{14,2}_{2,14}, e^{14,4}_{4,14}, e^{14,5}_{5,14}, e^{15,3}_{3,15}, e^{15,4}_{4,15}, e^{15,5}_{5,15} \end{matrix}\right\},$$

$$\Delta q^{4}\left\{+\overline{q}\left\{\begin{matrix} e^{3,14}_{15,2}, e^{4,13}_{14,3}, e^{5,12}_{13,4} \\ e^{15,2}_{3,14}, e^{14,3}_{4,13}, e^{13,4}_{5,12} \end{matrix}\right\}, -1\left\{\begin{matrix} e^{4,13}_{15,2}, e^{5,12}_{14,3} \\ e^{15,2}_{4,13}, e^{14,3}_{5,12} \end{matrix}\right\}, +q\left\{\begin{matrix} e^{5,12}_{15,2} \\ e^{15,2}_{5,12} \end{matrix}\right\}\right\},$$

$$\Delta q^{5}\left\{-\overline{q}\left\{\begin{matrix} e^{7,10}_{11,6}, e^{8,9}_{10,7}, e^{9,8}_{10,7} \\ e^{11,6}_{7,10}, e^{10,7}_{8,9}, e^{10,7}_{9,8} \end{matrix}\right\}, +1\left\{\begin{matrix} e^{8,9}_{11,6}, e^{9,8}_{11,6} \\ e^{11,6}_{8,9}, e^{11,6}_{9,8} \end{matrix}\right\}, -q\left\{\begin{matrix} e^{10,7}_{11,6} \\ e^{11,6}_{10,7} \end{matrix}\right\}\right\},$$

$$-\Delta q^{-\alpha+2}\left\{\begin{matrix} e^{3,7}_{9,2}, e^{3,8}_{10,2}, e^{4,6}_{7,3}, e^{4,8}_{11,2}, e^{4,10}_{11,3}, e^{5,6}_{8,3}, e^{5,7}_{8,4}, e^{5,9}_{10,4} \\ e^{9,2}_{3,7}, e^{10,2}_{3,8}, e^{7,3}_{4,6}, e^{11,2}_{4,8}, e^{11,3}_{4,10}, e^{8,3}_{5,6}, e^{8,4}_{5,7}, e^{10,4}_{5,9} \end{matrix}\right\},$$

$$-\Delta q^{\alpha+5}\left\{\begin{matrix} e^{7,13}_{14,6}, e^{8,12}_{13,7}, e^{9,13}_{15,6}, e^{9,14}_{15,7}, e^{10,12}_{13,9}, e^{10,14}_{15,8}, e^{11,12}_{14,9}, e^{11,13}_{14,10} \\ e^{14,6}_{7,13}, e^{13,7}_{8,12}, e^{15,6}_{9,13}, e^{15,7}_{9,14}, e^{13,9}_{10,12}, e^{15,8}_{10,14}, e^{14,9}_{11,12}, e^{14,10}_{11,13} \end{matrix}\right\},$$

$$\Delta q^{-\alpha+3}\left\{\begin{matrix} e^{4,6}_{9,2}, e^{5,6}_{10,2}, e^{5,7}_{11,2}, e^{5,9}_{11,3} \\ e^{9,2}_{4,6}, e^{10,2}_{5,6}, e^{11,2}_{5,7}, e^{11,3}_{5,9} \end{matrix}\right\}, \qquad \Delta q^{\alpha+6}\left\{\begin{matrix} e^{8,12}_{14,6}, e^{10,12}_{15,6}, e^{11,12}_{15,7}, e^{11,13}_{15,8} \\ e^{14,6}_{8,12}, e^{15,6}_{10,12}, e^{15,7}_{11,12}, e^{15,8}_{11,13} \end{matrix}\right\},$$

$$-\Delta q^{-2\alpha+\frac{1}{2}} A_0^{\frac{1}{2}} A_1^{\frac{1}{2}}\left\{\begin{matrix} e^{2,3}_{6,1}, e^{2,4}_{7,1}, e^{2,5}_{8,1}, e^{3,4}_{9,1}, e^{3,5}_{10,1}, e^{4,5}_{11,1} \\ e^{6,1}_{2,3}, e^{7,1}_{2,4}, e^{8,1}_{2,5}, e^{9,1}_{3,4}, e^{10,1}_{3,5}, e^{11,1}_{4,5} \end{matrix}\right\},$$

$$\Delta q^{-2\alpha+\frac{3}{2}} A_0^{\frac{1}{2}} A_1^{\frac{1}{2}}\left\{\begin{matrix} e^{3,2}_{6,1}, e^{4,2}_{7,1}, e^{5,2}_{8,1}, e^{4,3}_{9,1}, e^{5,3}_{10,1}, e^{5,4}_{11,1} \\ e^{6,1}_{3,2}, e^{7,1}_{4,2}, e^{8,1}_{5,2}, e^{9,1}_{4,3}, e^{10,1}_{5,3}, e^{11,1}_{5,4} \end{matrix}\right\},$$

$$\Delta q^{-\alpha+3} A_1 \left\{ \begin{array}{l} +\overline{q} \left\{ \begin{array}{l} e^{6,4}_{7,3}, e^{6,5}_{8,3}, e^{7,5}_{8,4}, e^{9,5}_{10,4} \\ e^{7,3}_{6,4}, e^{8,3}_{6,5}, e^{8,4}_{7,5}, e^{10,4}_{9,5} \end{array} \right\} \\ -1 \left\{ \begin{array}{l} e^{6,4}_{9,2}, e^{6,5}_{10,2}, e^{7,5}_{11,2}, e^{9,5}_{11,3} \\ e^{9,2}_{6,4}, e^{10,2}_{6,5}, e^{11,2}_{7,5}, e^{11,3}_{9,5} \end{array} \right\} \\ +q \left\{ \begin{array}{l} e^{7,3}_{9,2}, e^{8,3}_{10,2}, e^{8,4}_{11,2}, e^{10,4}_{11,3} \\ e^{9,2}_{7,3}, e^{10,2}_{8,3}, e^{11,2}_{8,4}, e^{11,3}_{10,4} \end{array} \right\} \end{array} \right\},$$

$$\Delta q^{\alpha+6} A_2 \left\{ \begin{array}{l} -\overline{q} \left\{ \begin{array}{l} e^{12,8}_{13,7}, e^{12,10}_{13,9}, e^{12,11}_{14,9}, e^{13,11}_{14,10} \\ e^{13,7}_{12,8}, e^{13,9}_{12,10}, e^{14,9}_{12,11}, e^{14,10}_{13,11} \end{array} \right\} \\ +1 \left\{ \begin{array}{l} e^{12,8}_{14,6}, e^{12,10}_{15,6}, e^{12,11}_{15,7}, e^{13,11}_{15,8} \\ e^{14,6}_{12,8}, e^{15,6}_{12,10}, e^{15,7}_{12,11}, e^{15,8}_{13,11} \end{array} \right\} \\ -q \left\{ \begin{array}{l} e^{13,7}_{14,6}, e^{13,9}_{15,6}, e^{14,9}_{15,7}, e^{14,10}_{15,8} \\ e^{14,6}_{13,7}, e^{15,6}_{13,9}, e^{15,7}_{14,9}, e^{15,8}_{14,10} \end{array} \right\} \end{array} \right\},$$

$$\Delta q^{-\alpha+2} A_0^{\frac{1}{2}} A_2^{\frac{1}{2}} \left\{ \begin{array}{l} -\overline{q} \left\{ \begin{array}{l} e^{2,9}_{12,1}, e^{2,10}_{13,1}, e^{2,11}_{14,1}, e^{3,11}_{15,1} \\ e^{12,1}_{2,9}, e^{13,1}_{2,10}, e^{14,1}_{2,11}, e^{15,1}_{3,11} \end{array} \right\} \\ +1 \left\{ \begin{array}{l} e^{3,7}_{12,1}, e^{3,8}_{13,1}, e^{4,8}_{14,1}, e^{4,10}_{15,1} \\ e^{12,1}_{3,7}, e^{13,1}_{3,8}, e^{14,1}_{4,8}, e^{15,1}_{4,10} \end{array} \right\} \\ -q \left\{ \begin{array}{l} e^{4,6}_{12,1}, e^{5,6}_{13,1}, e^{5,7}_{14,1}, e^{5,9}_{15,1} \\ e^{12,1}_{4,6}, e^{13,1}_{5,6}, e^{14,1}_{5,7}, e^{15,1}_{5,9} \end{array} \right\} \end{array} \right\},$$

$$\Delta q^{\alpha+5} A_1^{\frac{1}{2}} A_3^{\frac{1}{2}} \left\{ \begin{array}{l} +\overline{q} \left\{ \begin{array}{l} e^{6,14}_{16,2}, e^{6,15}_{16,3}, e^{7,15}_{16,4}, e^{8,15}_{16,5} \\ e^{16,2}_{6,14}, e^{16,3}_{6,15}, e^{16,4}_{7,15}, e^{16,5}_{8,15} \end{array} \right\} \\ -1 \left\{ \begin{array}{l} e^{7,13}_{16,2}, e^{9,13}_{16,3}, e^{9,14}_{16,4}, e^{10,14}_{16,5} \\ e^{16,2}_{7,13}, e^{16,3}_{9,13}, e^{16,4}_{9,14}, e^{16,5}_{10,14} \end{array} \right\} \\ +q \left\{ \begin{array}{l} e^{8,12}_{16,2}, e^{10,12}_{16,3}, e^{11,12}_{16,4}, e^{11,13}_{16,5} \\ e^{16,2}_{8,12}, e^{16,3}_{10,12}, e^{16,4}_{11,12}, e^{16,5}_{11,13} \end{array} \right\} \end{array} \right\},$$

$$\Delta^2 q^{-\alpha+3} A_0^{\frac{1}{2}} A_1 A_2^{\frac{1}{2}} \left\{ \begin{array}{l} +\overline{q} \left\{ \begin{array}{l} e^{6,4}_{12,1}, e^{6,5}_{13,1}, e^{7,5}_{14,1}, e^{9,5}_{15,1} \\ e^{12,1}_{6,4}, e^{13,1}_{6,5}, e^{14,1}_{7,5}, e^{15,1}_{9,5} \end{array} \right\} \\ -1 \left\{ \begin{array}{l} e^{7,3}_{12,1}, e^{8,3}_{13,1}, e^{8,4}_{14,1}, e^{10,4}_{15,1} \\ e^{12,1}_{7,3}, e^{13,1}_{8,3}, e^{14,1}_{8,4}, e^{15,1}_{10,4} \end{array} \right\} \\ +q \left\{ \begin{array}{l} e^{9,2}_{12,1}, e^{10,2}_{13,1}, e^{11,2}_{14,1}, e^{11,3}_{15,1} \\ e^{12,1}_{9,2}, e^{13,1}_{10,2}, e^{14,1}_{11,2}, e^{15,1}_{11,3} \end{array} \right\} \end{array} \right\},$$

$$\Delta^2 q^{\alpha+6} A_1^{\frac{1}{2}} A_2 A_3^{\frac{1}{2}} \left\{ \begin{array}{l} +\overline{q} \left\{ \begin{array}{l} e^{12,8}_{16,2}, e^{12,10}_{16,3}, e^{12,11}_{16,4}, e^{13,11}_{16,5} \\ e^{16,2}_{12,8}, e^{16,3}_{12,10}, e^{16,4}_{12,11}, e^{16,5}_{13,11} \end{array} \right\} \\ -1 \left\{ \begin{array}{l} e^{13,7}_{16,2}, e^{13,9}_{16,3}, e^{14,9}_{16,4}, e^{14,10}_{16,5} \\ e^{16,2}_{13,7}, e^{16,3}_{13,9}, e^{16,4}_{14,9}, e^{16,5}_{14,10} \end{array} \right\} \\ +q \left\{ \begin{array}{l} e^{14,6}_{16,2}, e^{15,6}_{16,3}, e^{15,7}_{16,4}, e^{15,8}_{16,5} \\ e^{16,2}_{14,6}, e^{16,3}_{15,6}, e^{16,4}_{15,7}, e^{16,5}_{15,8} \end{array} \right\} \end{array} \right\},$$

$$-\Delta q^{\frac{5}{2}} A_1^{\frac{1}{2}} A_2^{\frac{1}{2}} \begin{Bmatrix} e^{6,7}_{12,2}, e^{6,8}_{13,2}, e^{6,9}_{12,3}, e^{6,10}_{13,3}, e^{7,8}_{14,2}, e^{7,9}_{12,4}, e^{7,11}_{14,4}, e^{8,10}_{13,5}, e^{8,11}_{14,5}, e^{9,10}_{15,3}, e^{9,11}_{15,4}, e^{10,11}_{15,5} \\ e^{12,2}_{6,7}, e^{13,2}_{6,8}, e^{12,3}_{6,9}, e^{13,3}_{6,10}, e^{14,2}_{7,8}, e^{12,4}_{7,9}, e^{14,4}_{7,11}, e^{13,5}_{8,10}, e^{14,5}_{8,11}, e^{15,3}_{9,10}, e^{15,4}_{9,11}, e^{15,5}_{10,11} \end{Bmatrix},$$

$$\Delta q^{\frac{7}{2}} A_1^{\frac{1}{2}} A_2^{\frac{1}{2}} \begin{Bmatrix} e^{7,6}_{12,2}, e^{8,6}_{13,2}, e^{8,7}_{14,2}, e^{9,6}_{12,3}, e^{9,7}_{12,4}, e^{10,6}_{13,3}, e^{10,8}_{13,5}, e^{10,9}_{15,3}, e^{11,7}_{14,4}, e^{11,8}_{14,5}, e^{11,9}_{15,4}, e^{11,10}_{15,5} \\ e^{12,2}_{7,6}, e^{13,2}_{8,6}, e^{14,2}_{8,7}, e^{12,3}_{9,6}, e^{12,4}_{9,7}, e^{13,3}_{10,6}, e^{13,5}_{10,8}, e^{15,3}_{10,9}, e^{14,4}_{11,7}, e^{14,5}_{11,8}, e^{15,4}_{11,9}, e^{15,5}_{11,10} \end{Bmatrix},$$

$$\Delta q^4 A_1^{\frac{1}{2}} A_2^{\frac{1}{2}} \left\{ -\overline{q}^{\frac{1}{2}} \begin{Bmatrix} e^{6,11}_{14,3}, e^{7,10}_{13,4}, e^{8,9}_{12,5} \\ e^{14,3}_{6,11}, e^{13,4}_{7,10}, e^{12,5}_{8,9} \end{Bmatrix}, +q^{\frac{1}{2}} \begin{Bmatrix} e^{9,8}_{13,4}, e^{10,7}_{12,5}, e^{6,11}_{15,2} \\ e^{13,4}_{9,8}, e^{12,5}_{10,7}, e^{15,2}_{6,11} \end{Bmatrix} \right\},$$

$$\Delta q^6 A_1^{\frac{1}{2}} A_2^{\frac{1}{2}} \left\{ -\overline{q}^{\frac{1}{2}} \begin{Bmatrix} e^{9,8}_{14,3}, e^{7,10}_{15,2}, e^{11,6}_{12,5} \\ e^{14,3}_{9,8}, e^{15,2}_{7,10}, e^{12,5}_{11,6} \end{Bmatrix}, +q^{\frac{1}{2}} \begin{Bmatrix} e^{11,6}_{13,4}, e^{10,7}_{14,3}, e^{8,9}_{15,2} \\ e^{13,4}_{11,6}, e^{14,3}_{10,7}, e^{15,2}_{8,9} \end{Bmatrix} \right\},$$

$$\Delta^2 q^4 A_1^{\frac{1}{2}} A_2^{\frac{1}{2}} \left\{ +\overline{q}^{\frac{1}{2}} \begin{Bmatrix} e^{7,10}_{14,3}, e^{8,9}_{13,4} \\ e^{14,3}_{7,10}, e^{13,4}_{8,9} \end{Bmatrix}, -q^{\frac{1}{2}} \begin{Bmatrix} e^{8,9}_{14,3}, e^{10,7}_{13,4} \\ e^{13,4}_{10,7}, e^{14,3}_{8,9} \end{Bmatrix} \right\},$$

$$\Delta^2 q^{\frac{11}{2}} (\overline{q}^2 - q^2) A_1^{\frac{1}{2}} A_2^{\frac{1}{2}} \left\{ -\overline{q} \begin{Bmatrix} e^{9,8}_{15,2} \\ e^{15,2}_{9,8} \end{Bmatrix}, +1 \begin{Bmatrix} e^{10,7}_{15,2}, e^{11,6}_{14,3} \\ e^{15,2}_{10,7}, e^{14,3}_{11,6} \end{Bmatrix}, -q \begin{Bmatrix} e^{11,6}_{15,2} \\ e^{15,2}_{11,6} \end{Bmatrix} \right\},$$

$$\Delta^2 q^6 A_1 A_2 \left\{ \overline{q}^2 \begin{Bmatrix} e^{12,5}_{13,4} \\ e^{13,4}_{12,5} \end{Bmatrix}, -\overline{q} \begin{Bmatrix} e^{12,5}_{14,3} \\ e^{14,3}_{12,5} \end{Bmatrix}, 1 \begin{Bmatrix} e^{12,5}_{15,2}, e^{13,4}_{14,3} \\ e^{15,2}_{12,5}, e^{14,3}_{13,4} \end{Bmatrix}, -q \begin{Bmatrix} e^{13,4}_{15,2} \\ e^{15,2}_{13,4} \end{Bmatrix}, q^2 \begin{Bmatrix} e^{14,3}_{15,2} \\ e^{15,2}_{14,3} \end{Bmatrix} \right\},$$

$$\Delta^2 q^5 A_0^{\frac{1}{2}} A_1^{\frac{1}{2}} A_2^{\frac{1}{2}} A_3^{\frac{1}{2}} \times$$
$$\left\{ \overline{q}^2 \begin{Bmatrix} e^{6,11}_{16,1} \\ e^{16,1}_{6,11} \end{Bmatrix}, -\overline{q} \begin{Bmatrix} e^{7,10}_{16,1} \\ e^{16,1}_{7,10} \end{Bmatrix}, 1 \begin{Bmatrix} e^{8,9}_{16,1}, e^{9,8}_{16,1} \\ e^{16,1}_{8,9}, e^{16,1}_{9,8} \end{Bmatrix}, -q \begin{Bmatrix} e^{10,7}_{16,1} \\ e^{16,1}_{10,7} \end{Bmatrix}, q^2 \begin{Bmatrix} e^{11,6}_{16,1} \\ e^{16,1}_{11,6} \end{Bmatrix} \right\},$$

$$-\Delta q^3 A_0^{\frac{1}{2}} A_3^{\frac{1}{2}} \left\{ +\overline{q}^{\frac{3}{2}} \begin{Bmatrix} e^{2,15}_{16,1} \\ e^{16,1}_{2,15} \end{Bmatrix}, -\overline{q}^{\frac{1}{2}} \begin{Bmatrix} e^{3,14}_{16,1} \\ e^{16,1}_{3,14} \end{Bmatrix}, +q^{\frac{1}{2}} \begin{Bmatrix} e^{4,13}_{16,1} \\ e^{16,1}_{4,13} \end{Bmatrix}, -q^{\frac{3}{2}} \begin{Bmatrix} e^{5,12}_{16,1} \\ e^{16,1}_{5,12} \end{Bmatrix} \right\},$$

$$-\Delta^3 q^6 A_0^{\frac{1}{2}} A_1 A_2 A_3^{\frac{1}{2}} \left\{ +\overline{q}^{\frac{3}{2}} \begin{Bmatrix} e^{12,5}_{16,1} \\ e^{16,1}_{12,5} \end{Bmatrix}, -\overline{q}^{\frac{1}{2}} \begin{Bmatrix} e^{13,4}_{16,1} \\ e^{16,1}_{13,4} \end{Bmatrix}, +q^{\frac{1}{2}} \begin{Bmatrix} e^{14,3}_{16,1} \\ e^{16,1}_{14,3} \end{Bmatrix}, -q^{\frac{3}{2}} \begin{Bmatrix} e^{15,2}_{16,1} \\ e^{16,1}_{15,2} \end{Bmatrix} \right\},$$

$$-\Delta q^{2\alpha+\frac{13}{2}} A_2^{\frac{1}{2}} A_3^{\frac{1}{2}} \begin{Bmatrix} e^{12,13}_{16,6}, e^{12,14}_{16,7}, e^{13,14}_{16,8}, e^{12,15}_{16,9}, e^{13,15}_{16,10}, e^{14,15}_{16,11} \\ e^{16,6}_{12,13}, e^{16,7}_{12,14}, e^{16,8}_{13,14}, e^{16,9}_{12,15}, e^{16,10}_{13,15}, e^{16,11}_{14,15} \end{Bmatrix},$$

$$\Delta q^{2\alpha+\frac{15}{2}} A_2^{\frac{1}{2}} A_3^{\frac{1}{2}} \begin{Bmatrix} e^{13,12}_{16,6}, e^{14,12}_{16,7}, e^{14,13}_{16,8}, e^{15,12}_{16,9}, e^{15,13}_{16,10}, e^{15,14}_{16,11} \\ e^{16,6}_{13,12}, e^{16,7}_{14,12}, e^{16,8}_{14,13}, e^{16,9}_{15,12}, e^{16,10}_{15,13}, e^{16,11}_{15,14} \end{Bmatrix},$$

and the left handle C has 16 diagonal components:

$$C = q^{-4\alpha-6} \left\{ \begin{array}{l} +q^6 \left\{ e^1_1 \right\} \\ -q^3 \left\{ q^3 e^2_2,\ \ q e^3_3,\ \ \overline{q} e^4_4,\ \ \overline{q}^3 e^5_5 \right\} \\ + \left\{ q^4 e^6_6,\ \ q^2 e^7_7,\ \ e^8_8,\ \ e^9_9,\ \ \overline{q}^2 e^{10}_{10},\ \ \overline{q}^4 e^{11}_{11} \right\} \\ -\overline{q}^3 \left\{ \overline{q}^3 e^{15}_{15},\ \ \overline{q} e^{14}_{14},\ \ q e^{13}_{13},\ \ q^3 e^{12}_{12} \right\} \\ +\overline{q}^6 \left\{ e^{16}_{16} \right\} \end{array} \right\},$$

using the scaling factors:

$$\kappa_\sigma = q^{-4\alpha}, \qquad \kappa_C = q^{-4\alpha}.$$

B Evaluations of LG^3 and LG^4

Below, we present evaluations for LG^3 and LG^4 for a few 2 and 3-braid links. We use the same naming conventions for links as those of [4], except that we denote by 2^2_{1a} and 2^2_{1b} the 2 component links determined respectively by the braids $\sigma_1^{\pm 1}$.

To present evaluations of LG^m, we use a similar convention to that of [4]. The expression $s_0(A_0(q)), s_1(A_1(q)), \ldots, s_r(A_r(q))$, where the s_i are signs $\pm$ and the $A_i(q)$ are integer-coefficient Laurent polynomials in q, is intended to be read:

$$s_0(A_0(q)) + s_1(\overline{P}^1 + P^1)(A_1(q)) + \ldots + s_r(\overline{P}^r + P^r)(A_r(q)).$$

In these expressions $(A_i(q))$ is only a list of terms of $A_i(q)$ rather than an explicit sum, viz we have written $(x_1, x_2, \ldots, x_s)$ for $(x_1 + x_2 + \cdots + x_s)$. Recall that, for $LG^{m,n}$ (fixing m, n), we are using the variable $P = p^2 = q^{2\alpha+m-n}$; viz for $LG^3 \equiv LG^{3,1}$ we use $P = q^{2\alpha+2}$, and for $LG^4 \equiv LG^{4,1}$ we use $P = q^{2\alpha+3}$. For multicomponent links, if the polynomial is not invariant under $P \mapsto \overline{P}$, we write it out in full. This situation only occurs here for LG^3 for links of 2 components.

Within the q-polynomials, the same general behaviours of the coefficients as reported for LG^2 in [4] are seen. These calculations were performed on `SUN Ultra 60` UNIX based workstations, with a main memory of 256Mb, and the larger calculations sometimes used all of this memory.

Evaluations of LG^3

$$LG^3_{3_1} = -(q^2, 2q^4, 3q^6, q^8), +(q^2, 2q^4, 2q^6, q^8), -(q^4, q^6, q^8), +(q^6)$$

$$LG^3_{4_1} = +(5\overline{q}^4, 9\overline{q}^2, 17, 9q^2, 5q^4), -(2\overline{q}^4, 8\overline{q}^2, 10, 8q^2, 2q^4), +(3\overline{q}^2, 3, 3q^2), -(1)$$

$$\begin{aligned} LG^3_{5_1} = {} & +(q^4, 4q^6, 5q^8, 5q^{10}, 3q^{12}, q^{14}), -(q^4, 3q^6, 5q^8, 5q^{10}, 3q^{12}, q^{14}), \\ & +(q^4, 2q^6, 4q^8, 4q^{10}, 3q^{12}, q^{14}), -(q^6, 2q^8, 3q^{10}, 3q^{12}, q^{14}), \\ & +(q^8, 2q^{10}, 2q^{12}, q^{14}), -(q^{10}, q^{12}, q^{14}), +(q^{12}) \end{aligned}$$

$$\begin{aligned} LG^3_{5_2} = {} & -(7q^2, 17q^4, 32q^6, 25q^8, 15q^{10}, 3q^{12}), +(4q^2, 15q^4, 23q^6, 22q^8, 11q^{10}, 3q^{12}), \\ & -(7q^4, 10q^6, 12q^8, 5q^{10}, 2q^{12}), +(4q^6, 2q^8, 2q^{10}) \end{aligned}$$

$$\begin{aligned} LG^3_{6_2} = {} & -(7\overline{q}^2, 29, 60q^2, 74q^4, 60q^6, 26q^8, 5q^{10}), \\ & +(6\overline{q}^2, 24, 52q^2, 67q^4, 54q^6, 26q^8, 5q^{10}), \\ & -(2\overline{q}^2, 14, 32q^2, 48q^4, 41q^6, 23q^8, 5q^{10}), +(4, 15q^2, 24q^4, 28q^6, 14q^8, 5q^{10}), \\ & -(4q^2, 10q^4, 12q^6, 8q^8, 2q^{10}), +(3q^4, 3q^6, 3q^8), -(q^6) \end{aligned}$$

$$\begin{aligned} LG^3_{6_3} = {} & +(9\overline{q}^6, 52\overline{q}^4, 106\overline{q}^2, 145, 106q^2, 52q^4, 9q^6), \\ & -(9\overline{q}^6, 42\overline{q}^4, 96\overline{q}^2, 120, 96q^2, 42q^4, 9q^6), \\ & +(6\overline{q}^6, 26\overline{q}^4, 63\overline{q}^2, 77, 63q^2, 26q^4, 6q^6), \\ & -(2\overline{q}^6, 13\overline{q}^4, 28\overline{q}^2, 40, 28q^2, 13q^4, 2q^6), +(4\overline{q}^4, 10\overline{q}^2, 14, 10q^2, 4q^4), \\ & -(3\overline{q}^2, 3, 3q^2), +(1) \end{aligned}$$

$$LG^3_{2^2_{1a}} = -(\overline{p}^1 - p^1)(q, q^3, q^5), +(\overline{p}^3 - p^3)(q^3)$$

$$LG^3_{2^2_{1b}} = +(\overline{p}^1 - p^1)(\overline{q}^5, \overline{q}^3, \overline{q}^1), -(\overline{p}^3 - p^3)(\overline{q}^3)$$

$$\begin{aligned} LG^3_{4^2_{1a}} = &+(\overline{p}^1 - p^1)(q^3, 3q^5, 4q^7, 3q^9, q^{11}), -(\overline{p}^3 - p^3)(q^3, 2q^5, 3q^7, 3q^9, q^{11}) \\ &+(\overline{p}^5 - p^5)(q^5, 2q^7, 2q^9, q^{11}), -(\overline{p}^7 - p^7)(q^7, q^9, q^{11}), +(\overline{p}^9 - p^9)(q^9) \end{aligned}$$

$$LG^3_{4^2_{1b}} = -(\overline{p}^1 - p^1)(4q^1, 6q^3, 8q^5, 4q^7, 2q^9), +(\overline{p}^3 - p^3)(4q^3, 2q^5, 2q^7)$$

$$\begin{aligned} LG^3_{5^2_1} = &+(\overline{p}^1 - p^1)(5\overline{q}^7, 23\overline{q}^5, 36\overline{q}^3, 37\overline{q}^1, 19q^1, 6q^3), \\ &-(\overline{p}^3 - p^3)(5\overline{q}^7, 14\overline{q}^5, 28\overline{q}^3, 22\overline{q}^1, 13q^1, 2q^3) \\ &+(\overline{p}^5 - p^5)(2\overline{q}^7, 8\overline{q}^5, 12\overline{q}^3, 10\overline{q}^1, 4q^1) \\ &-(\overline{p}^7 - p^7)(3\overline{q}^5, 3\overline{q}^3, 3\overline{q}^1), +(\overline{p}^9 - p^9)(\overline{q}^3) \end{aligned}$$

$$\begin{aligned} LG^3_{6^2_1} = &-(\overline{p}^1 - p^1)(q^5, 4q^7, 6q^9, 7q^{11}, 5q^{13}, 3q^{15}, q^{17}) \\ &+(\overline{p}^3 - p^3)(q^5, 3q^7, 6q^9, 6q^{11}, 5q^{13}, 3q^{15}, q^{17}) \\ &-(\overline{p}^5 - p^5)(q^5, 2q^7, 4q^9, 5q^{11}, 5q^{13}, 3q^{15}, q^{17}) \\ &+(\overline{p}^7 - p^7)(q^7, 2q^9, 4q^{11}, 4q^{13}, 3q^{15}, q^{17}) \\ &-(\overline{p}^9 - p^9)(q^9, 2q^{11}, 3q^{13}, 3q^{15}, q^{17}), +(\overline{p}^{11} - p^{11})(q^{11}, 2q^{13}, 2q^{15}, q^{17}) \\ &-(\overline{p}^{13} - p^{13})(q^{13}, q^{15}, q^{17}), +(\overline{p}^{15} - p^{15})(q^{15}) \end{aligned}$$

$$\begin{aligned} LG^3_{6^2_2} = &+(\overline{p}^1 - p^1)(7q^3, 26q^5, 52q^7, 59q^9, 43q^{11}, 17q^{13}, 3q^{15}) \\ &-(\overline{p}^3 - p^3)(4q^3, 18q^5, 36q^7, 48q^9, 34q^{11}, 16q^{13}, 3q^{15}) \\ &+(\overline{p}^5 - p^5)(7q^5, 19q^7, 27q^9, 23q^{11}, 11q^{13}, 3q^{15}) \\ &-(\overline{p}^7 - p^7)(7q^7, 10q^9, 12q^{11}, 5q^{13}, 2q^{15}), +(\overline{p}^9 - p^9)(4q^9, 2q^{11}, 2q^{13}) \end{aligned}$$

$$\begin{aligned} LG^3_{6^3_1} = &-(10\overline{q}^2, 38, 82q^2, 100q^4, 84q^6, 36q^8, 8q^{10}), \\ &+(7\overline{q}^2, 32, 69q^2, 91q^4, 73q^6, 36q^8, 7q^{10}), \\ &-(2\overline{q}^2, 18, 42q^2, 63q^4, 53q^6, 29q^8, 6q^{10}), +(5, 19q^2, 30q^4, 33q^6, 16q^8, 5q^{10}), \\ &-(5q^2, 11q^4, 13q^6, 8q^8, 2q^{10}), +(3q^4, 3q^6, 3q^8), -(q^6) \end{aligned}$$

$$\begin{aligned} LG^3_{6^3_2} = &+(18\overline{q}^6, 100\overline{q}^4, 206\overline{q}^2, 276, 206q^2, 100q^4, 18q^6), \\ &-(17\overline{q}^6, 81\overline{q}^4, 183\overline{q}^2, 230, 183q^2, 81q^4, 17q^6), \\ &+(11\overline{q}^6, 48\overline{q}^4, 117\overline{q}^2, 143, 117q^2, 48q^4, 11q^6), \\ &-(3\overline{q}^6, 23\overline{q}^4, 49\overline{q}^2, 70, 49q^2, 23q^4, 3q^6), +(6\overline{q}^4, 16\overline{q}^2, 22, 16q^2, 6q^4), \\ &-(4\overline{q}^2, 4, 4q^2), +(1) \end{aligned}$$

$$\begin{aligned} LG^3_{6^3_3} = &+(2q^6, 2q^8, 2q^{10}), -(q^4, 2q^6, 3q^8, 2q^{10}, q^{12}), +(q^4, 2q^6, 3q^8, 2q^{10}, q^{12}), \\ &-(q^6, q^8, q^{10}, q^{12}), +(q^{10}, q^{12}, q^{14}), -(q^{10}, q^{12}, q^{14}), +(q^{12}) \end{aligned}$$

Evaluations of LG^4

$$LG^4_{2^2_{1a}} = +(q^2, q^4, 2q^6, q^8, q^{10}), -(q^3, q^5, q^7, q^9), +(q^6)$$

$$LG^4_{2^2_{1b}} = +(\overline{q}^{10}, \overline{q}^8, 2\overline{q}^6, \overline{q}^4, \overline{q}^2), -(\overline{q}^9, \overline{q}^7, \overline{q}^5, \overline{q}^3), +(\overline{q}^6)$$

$$\begin{aligned} LG^4_{3_1} = {} & +(q^4, 2q^6, 4q^8, 4q^{10}, 5q^{12}, 2q^{14}, q^{16}), -(q^5, 2q^7, 4q^9, 4q^{11}, 3q^{13}, 2q^{15}), \\ & +(q^6, 2q^8, 2q^{10}, 3q^{12}, q^{14}, q^{16}), -(q^9, q^{11}, q^{13}, q^{15}), +(q^{12}) \end{aligned}$$

$$\begin{aligned} LG^4_{4^2_{1a}} = {} & +(q^6, 2q^8, 5q^{10}, 8q^{12}, 10q^{14}, 8q^{16}, 7q^{18}, 2q^{20}, q^{22}), \\ & -(q^7, 3q^9, 6q^{11}, 8q^{13}, 9q^{15}, 7q^{17}, 4q^{19}, 2q^{21}), \\ & +(q^8, 3q^{10}, 5q^{12}, 7q^{14}, 6q^{16}, 6q^{18}, 2q^{20}, q^{22}), \\ & -(q^9, 2q^{11}, 3q^{13}, 5q^{15}, 4q^{17}, 3q^{19}, 2q^{21}), +(q^{12}, 2q^{14}, 2q^{16}, 3q^{18}, q^{20}, q^{22}), \\ & -(q^{15}, q^{17}, q^{19}, q^{21}), +(q^{18}) \end{aligned}$$

$$\begin{aligned} LG^4_{5_1} = {} & +(q^8, 2q^{10}, 6q^{12}, 10q^{14}, 15q^{16}, 16q^{18}, 15q^{20}, 10q^{22}, 7q^{24}, 2q^{26}, q^{28}), \\ & -(q^9, 3q^{11}, 7q^{13}, 12q^{15}, 15q^{17}, 15q^{19}, 13q^{21}, 8q^{23}, 4q^{25}, 2q^{27}), \\ & +(q^{10}, 4q^{12}, 7q^{14}, 11q^{16}, 13q^{18}, 13q^{20}, 9q^{22}, 7q^{24}, 2q^{26}, q^{28}), \\ & -(q^{11}, 3q^{13}, 6q^{15}, 9q^{17}, 10q^{19}, 10q^{21}, 7q^{23}, 4q^{25}, 2q^{27}), \\ & +(q^{12}, 2q^{14}, 4q^{16}, 6q^{18}, 7q^{20}, 6q^{22}, 6q^{24}, 2q^{26}, q^{28}), \\ & -(q^{15}, 2q^{17}, 3q^{19}, 5q^{21}, 4q^{23}, 3q^{25}, 2q^{27}), \\ & +(q^{18}, 2q^{20}, 2q^{22}, 3q^{24}, q^{26}, q^{28}), -(q^{21}, q^{23}, q^{25}, q^{27}), +(q^{24}) \end{aligned}$$

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